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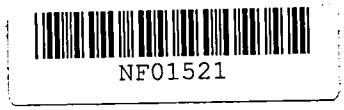
K.B. Subrahmanyam and K.R.V. Kaza
Lewis Research Center
Cleveland, Ohio

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INFLUENCE OF THIRD-DEGREE GEOMETRIC NONLINEARITIES ON THE VIBRATION AND STABILITY OF PRETWISTED, PRECONED, ROTATING BLADES

K.B. Subrahmanyam* and K.R.V. Kaza
National Aeronautics and Space Administration
Lewis Research Center
Cleveland, Ohio 44135

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SUMMARY

This report is concerned with the study of the influence of third degree geometric nonlinear terms on the vibration and stability characteristics of rotating, pretwisted, and preconed blades. The governing coupled flapwise bending, edgewise bending, and torsional equations are derived including third-degree geometric nonlinear elastic terms by making use of the geometric nonlinear theory of elasticity in which the elongations and shears are negligible compared to unity. These equations are specialized for blades of doubly symmetric cross section with linear variation of pretwist over the blade length. The nonlinear steady state equations and the linearized perturbation equations are solved by using the Galerkin method, and by utilizing the nonrotating normal modes for the shape functions. Parametric results obtained for various cases of rotating blades from the present theoretical formulation are compared to those produced from the finite element code MSC/NASTRAN, and also to those produced from an in-house experimental test rig. It is shown that the spurious instabilities, observed for thin, rotating blades when second degree geometric nonlinearities are used, can be eliminated by including the third-degree elastic nonlinear terms. Furthermore, inclusion of third degree terms improves the correlation between the theory and experiment.

INTRODUCTION

Considerable work has been done in the area of helicopter rotor blade dynamics to date. Several investigators have derived the governing equations of motion for the helicopter rotor blades incorporating various degrees of complexity (refs. 1 to 3). It is now established beyond any doubt that geometric nonlinearities must be included in the analysis for a fair prediction of rotating blade frequencies and stability boundaries. However, there remain certain questions concerning the degree to which the geometric nonlinearities should be retained, and concerning the initial assumptions in prescribing an ordering scheme (refs. 4 and 5).

*On leave from NBKR Institute of Science and Technology, Mechanical Engineering Department, Vidyanagar 524413, India, and presently Senior Research Associate, University of Toledo, Mechanical Engineering Department, Toledo, Ohio 43606.

Another area, somewhat similar to the helicopter rotor blade dynamics but further complicated due to the geometry, is the advanced turboprop blade dynamics (refs. 6 and 7). Unlike helicopter rotor blades, the turboprop blades are more shell-like, possess variable sweep (a component of which may be considered as blade precone and typically of the order of 50° at blade tip for the turboprop blade) along their span, and are subjected to considerable rotational speeds which can cause large steady state deformations. Because of the complex geometry of the advanced turboprop blades, finite element methods are well suited for their vibration analysis. However, when finite element methods are used, it is not practical to conduct parametric studies to obtain a physical insight into the various complicating effects.

An investigation was therefore undertaken by the present authors to assess the individual and combined influence of such complicating effects as sweep, Coriolis forces, pretwist, geometric nonlinearities, and rotation on the coupled frequencies for blade configurations that are representative of advanced turboprop blades. The blades were modelled as beams having constant, but large, precone and linear pretwist. A preliminary study of this beam model for torsionally rigid blades (refs. 8 and 9) showed that both the linear and nonlinear Coriolis forces must be retained in analyzing thick blades while the Coriolis effects can safely be ignored in analyzing thin blades. For a fair prediction of stability boundaries however, inclusion of the torsional degree of freedom was found necessary. Thus, the coupled flapwise bending, edgewise bending, torsion, and extension equations were addressed in reference 10. It was shown that for small precones and moderate rotational speeds, the frequencies produced by the beam theory with geometric nonlinearities up through second degree are in excellent agreement with those produced by MSC/NASTRAN. However, for large precones and large rotational speeds, the beam theory with second degree geometric nonlinearities was found to be inadequate generally, particularly so in the case of thin blades for which an early torsional divergence was noticed in comparison to MSC/NASTRAN predictions. It may be stated that the beam theory used in reference 10 and also the finite element code MSC/NASTRAN, use the geometric nonlinear theory of elasticity in which the deflections are treated as large and the strains as small. However, the nonlinear terms greater than second degree were explicitly discarded in the beam theory used in reference 10, while no such simplification is implicit in MSC/NASTRAN.

Since it has been shown in reference 10 that the addition of second degree elastic terms, often referred to as Mil's terms (ref. 11), is responsible for creating an early torsional divergence in the case of thin blades, it is proposed to derive the appropriate third degree elastic terms in the coupled flapwise bending, edgewise bending and torsion equations of pretwisted, preconed blades. By introducing the additional, third degree, geometric nonlinear terms into the solution procedure outlined in reference 10, it is proposed to establish that the spurious instabilities can be removed, and that reliable steady state deflections and frequencies can be obtained for complex blade configurations. Finally, it is proposed to present and discuss the effects of second- and third-degree geometric nonlinearities on thin and thick blades, and to establish the limits of reliability of the present beam theory calculations.

EQUATIONS OF MOTION AND METHOD OF SOLUTION

Figure 1 shows a linearly pretwisted, preconed, and rotating blade of uniform rectangular cross section. The coupled flapwise bending, edgewise bending and torsional equations of motion for such a blade with large precone were derived in reference 10 including second degree geometric nonlinearities and Coriolis effects. By including the additional elastic terms up through third degree geometric nonlinearity as given in appendix A, the resulting final equations can be shown to be of the following form (a list of notation is given in appendix C):

Flapwise bending:

$$
\begin{aligned}
&m\ddot{w} + 2m\Omega \sin \beta_{pc} \dot{v} - (Iw')' - m\Omega^2 \sin^2 \beta_{pc} w \\
&+ m\Omega^2 \left(k_{m2}^2 - k_{m1}^2\right) \cos 2\beta_{pc} (v' \sin \theta \cos \theta)' \\
&+ m\Omega^2 \cos 2\beta_{pc} \left[w' \left(k_{m2}^2 \sin^2 \theta + k_{m1}^2 \cos^2 \theta\right)\right]' \\
&+ m\Omega^2 \left(k_{m2}^2 - k_{m1}^2\right) \sin \beta_{pc} \cos \beta_{pc} (\phi \sin 2\theta)' \\
&+ 2\Omega \cos \beta_{pc} \left[\dot{\phi} \left(mk_{m2}^2 \sin^2 \theta + mk_{m1}^2 \cos^2 \theta\right)\right]' \\
&+ \left\{w'' \left[EI_{\eta\eta} \cos^2 \theta + EI_{\xi\xi} \sin^2 \theta + \phi \left(EI_{\xi\xi} - EI_{\eta\eta}\right) \sin 2\theta\right]\right. \\
&\left. + v'' \left[(EI_{\xi\xi} - EI_{\eta\eta})(\sin \theta \cos \theta + \phi \cos 2\theta)\right] - \phi' v' GJ \right\}'' + \left\{H_1\right\} \\
&= - m\Omega^2 \sin \beta_{pc} \cos \beta_{pc} (x - u_F) - \Omega^2 \sin \beta_{pc} \cos \beta_{pc} \left(mk_{m2}^2 \sin^2 \theta + mk_{m1}^2 \cos^2 \theta\right)'
\end{aligned}
\tag{1}
$$

Edgewise bending:

$$m\ddot{v} - 2m\Omega \sin \beta_{pc} \dot{w} - m\Omega^2 v + (\phi' w'' GJ)' - 2m\Omega \cos \beta_{pc} \dot{u}_F - (Tv')'$$

$$+ 2\Omega \cos \beta_{pc} \left[\dot{\phi}(mk_{m2}^2 - mk_{m1}^2) \sin \theta \cos \theta\right]'$$

$$+ m\Omega^2 \left(k_{m2}^2 - k_{m1}^2\right) \cos 2\beta_{pc} (w' \sin \theta \cos \theta)'$$

$$+ m\Omega^2 \left(k_{m2}^2 - k_{m1}^2\right) \sin \beta_{pc} \cos \beta_{pc} (\phi \cos 2\theta)'$$

$$- m\Omega^2 \sin^2 \beta_{pc} \; v' \left[\left(k_{m2}^2 \cos^2 \theta + k_{m1}^2 \sin^2 \theta\right)\right]'$$

$$+ \left\{w'' \left[(EI_{\xi\xi} - EI_{\eta\eta})(\sin \theta \cos \theta + \phi \cos 2\theta)\right]\right.$$

$$\left. + v'' \left[EI_{\eta\eta}(\sin^2 \theta + \phi \sin 2\theta) + EI_{\xi\xi} (\cos^2 \theta - \phi \sin 2\theta)\right]\right\}'' + \{H_2\}$$

$$= -\Omega^2 \left[\sin \beta_{pc} \cos \beta_{pc} \left(mk_{m2}^2 - mk_{m1}^2\right) \sin \theta \cos \theta\right]' \qquad (2)$$

Torsion:

$$mk_m^2 \ddot{\phi} + m\Omega^2 \phi \cos^2 \beta_{pc} \left(k_{m2}^2 - k_{m1}^2\right) \cos 2\theta$$

$$+ 2m\Omega \cos \beta_{pc} \left\{\left(k_{m2}^2 - k_{m1}^2\right) \dot{v}' \sin \theta \cos \theta\right.$$

$$\left. + \dot{w}' \left(k_{m2}^2 \sin^2 \theta + k_{m1}^2 \cos^2 \theta\right)\right\} + \left(EC_1 \phi''\right)'' - \left[EAk_A^2 u' (\theta_{pt}' + \phi')\right.$$

$$\left. + EB_1 \theta_{pt}'^2 \phi' + GJ\phi' - v' w'' GJ\right]' + \left(mk_\lambda^4 \Omega^2 \cos^2 \beta_{pc} \phi'\right)'$$

$$+ (EI_{\xi\xi} - EI_{\eta\eta}) \left[v'' w'' \cos 2\theta + (w''^2 - v''^2) \sin \theta \cos \theta\right]$$

$$- m\Omega^2 \sin \beta_{pc} \cos \beta_{pc} \left(k_{m2}^2 - k_{m1}^2\right) v' \cos 2\theta$$

$$- m\Omega^2 \sin \beta_{pc} \cos \beta_{pc} \left(k_{m2}^2 - k_{m1}^2\right) w' \sin 2\theta + \{H_3\}$$

$$= - m\Omega^2 \cos^2 \beta_{pc} \left(k_{m2}^2 - k_{m1}^2\right) \sin \theta \cos \theta \qquad (3)$$

Where:

$$T = -\int_x^L m\Big[\Omega^2 w \sin\beta_{pc}\cos\beta_{pc} - 2\Omega\dot{v}\cos\beta_{pc} - \Omega^2(R + x - u_F)\cos^2\beta_{pc} - \ddot{u}_F\Big]\,dx \tag{4}$$

$$u_F = \frac{1}{2}\int_0^x (v'^2 + w'^2)\,dx \tag{5}$$

$$EAu' = T - EA\left[k_A^2\,\phi'\theta'_{pt}\right] \tag{6}$$

and

$$\left.\begin{aligned} &m = \iint \rho dydz,\ A = \iint dydz,\ I_{\xi\xi} = \iint y^2 dydz,\ I_{\eta\eta} = \iint z^2 dydz,\\ &Ak_A^2 = \iint (y^2 + z^2)\,dydz,\ B_1 = \iint (y^2 + z^2)^2 dydz,\ C_1 = \iint \lambda^2 dydz,\\ &J = \iint \Big\{(y - \lambda_z)^2 + (z + \lambda_y)^2\Big\} dydz,\ mk_{m1}^2 = \iint \rho z^2 dydz,\\ &mk_{m2}^2 = \iint \rho y^2 dydz,\ k_m^2 = k_{m1}^2 + k_{m2}^2,\ ()' = \frac{\partial}{\partial x}(),\ \lambda_y = \frac{\partial\lambda}{\partial y},\ \lambda_z = \frac{\partial\lambda}{\partial z},\ \dot{()} = \frac{\partial}{\partial t}(). \end{aligned}\right\} \tag{7}$$

It should be noted here that in writing equations (1) to (3), the geometric pitch angle θ_{pt} is replaced by θ for the sake of convenience in writing the equations. This definition of $\theta(= \alpha + \gamma\eta)$ in the main body of the text should not be confused with the different definition for the same parameter used in appendix A while deriving the curvatures and additional nonlinear terms. Furthermore, the additional third degree geometric nonlinear terms of elastic origin, represented by H_1, H_2, and H_3 and in the flatwise, edgewise, and torsion equations, are given in appendix A. In the absence of these additional third degree geometric nonlinear terms, equations (1) and (2) contain terms up to $O(\epsilon^4)$ in the elastic forces (see ref. 3 for the ordering scheme followed for various parameters. For example, $\bar{w}$, $\bar{v}$, ϕ are of the order $O(\epsilon)$; $\bar{w}'\phi$ is of the order $O(\epsilon^2)$ etc.), and $O(\epsilon^2)$ in the inertial forces, while equation (3), contains elastic force terms up to the order $O(\epsilon^5)$ and inertial force terms up to the order $O(\epsilon^3)$. It was shown in reference 10 that the addition of any further higher order terms, either linear or second degree geometrically nonlinear, does not alter the quality of the final results. Thus, when the present third degree geometric nonlinear elastic terms are added to the second degree equations of reference 10, it is obvious that the order of the elastic terms in the bending and torsion equations has increased by at least one order of magnitude. For consistency of retaining the terms of appropriate magnitudes in the equations, one should, in principle, retain several other elastic terms discarded formerly in the second degree equations. However in view of the observations made in reference 10, the terms other than those

retained in equations (1) to (3) produce negligible variations on the frequencies and stability boundaries, and therefore are discarded even in the presence of the third degree geometric nonlinear terms.

Equations (1) to (3) can be written in nondimensional forms by defining the following relations:

$$\bar{w} = \frac{W}{L}, \quad \bar{v} = \frac{V}{L}, \quad \eta = \frac{X}{L}, \quad \tau = \Omega t, \quad \bar{R} = \frac{R}{L} \quad \text{etc.} \tag{8}$$

and

$$\frac{d}{dx} = \frac{1}{L}\frac{d}{d\eta}, \quad \frac{d}{dt} = \Omega \frac{d}{d\tau} \quad \text{etc.} \tag{9}$$

The resulting nondimensional equations are given below.

$$\begin{aligned}
&\ddot{\bar{w}} + 2 \sin \beta_{pc} \dot{\bar{v}} - \bar{w} \sin^2 \beta_{pc} + \left[2 \cos \beta_{pc} \dot{\phi} \left(mk_{m2}^2 \sin^2 \theta + mk_{m1}^2 \cos^2 \theta\right)\right] \Big/ mL^2 \\
&- \frac{1}{2} \sin \beta_{pc} \cos \beta_{pc} \int_0^{\eta} (\bar{v}'^2 + \bar{w}')^2 d\eta \\
&- \cos^2 \beta_{pc} (\bar{w}''Q - \bar{w}'s) - 2 \cos \beta_{pc} \left[\bar{w}'' \int_{\eta}^{1} \dot{\bar{v}}\, d\eta - \bar{w}'\dot{\bar{v}}\right] \\
&+ \sin \beta_{pc} \cos \beta_{pc} \left[\bar{w}'' \int_{\eta}^{1} \bar{w}\, d\eta - \bar{w}'\bar{w}\right] + \xi \bar{w}^{iv} \left(\cos^2 \theta + \frac{b^2}{d^2} \sin^2 \theta\right) \\
&+ \bar{w}''' (2\gamma\xi \sin 2\theta)\left(\frac{b^2}{d^2} - 1\right) + \bar{w}'' 2\gamma^2 \xi \cos 2\theta \left(\frac{b^2}{d^2} - 1\right) \\
&+ \xi \bar{v}^{iv} \left(\frac{b^2}{d^2} - 1\right) \sin \theta \cos \theta + \bar{v}'''(2\gamma\xi \cos 2\theta) \left(\frac{b^2}{d^2} - 1\right) \\
&- \bar{v}'' (2\gamma^2 \xi \sin 2\theta) \left(\frac{b^2}{d^2} - 1\right) + \xi \left(\frac{b^2}{d^2} - 1\right) \Big\{ \bar{w}^{iv}\phi \sin 2\theta \\
&+ 2\bar{w}'''\phi' \sin 2\theta + \bar{w}''\phi'' \sin 2\theta + 4\gamma\bar{w}'''\phi \cos 2\theta + 4\gamma\bar{w}''\phi' \cos 2\theta \\
&- 4\gamma^2 \bar{w}''\phi \sin 2\theta + \bar{v}^{iv}\phi \cos 2\theta + 2\bar{v}'''\phi' \cos 2\theta + \bar{v}''\phi'' \cos 2\theta \\
&- 4\gamma\bar{v}'''\phi \sin 2\theta - 4\gamma\bar{v}''\phi' \sin 2\theta - 4\gamma^2 \bar{v}''\phi \cos 2\theta \Big\} \\
&- \frac{GJ}{m\Omega^2 L^4} \{\phi'\bar{v}''' + 2\phi''\bar{v}'' + \phi'''\bar{v}'\} + \{\bar{H}_1\} \\
&= - \eta \sin \beta_{pc} \cos \beta_{pc} - \frac{\gamma I_{\eta\eta}}{AL^2} \left(\frac{b^2}{d^2} - 1\right) \sin \beta_{pc} \cos \beta_{pc} \sin 2\theta
\end{aligned} \tag{10}$$

$$\ddot{\bar{v}} - 2 \sin \beta_{pc} \dot{\bar{w}} - \bar{v} + \left[2 \cos \beta_{pc} \dot{\phi} \left(m k_{m2}^2 - m k_{m1}^2 \right) \sin \theta \cos \theta \right]^1 / m L^2$$

$$- \cos^2 \beta_{pc} \left(\bar{v}'' Q - \bar{v}' S \right) - 2 \cos \beta_{pc} \left[\bar{v}'' \int_\eta^1 \dot{\bar{v}} \, d\eta - \bar{v}' \dot{\bar{v}} \right]$$

$$- \cos \beta_{pc} \frac{d}{d\tau} \int_0^\eta \left(\bar{v}'^2 + \bar{w}'^2 \right) d\eta + \sin \beta_{pc} \cos \beta_{pc} \left[\bar{v}'' \int_\eta^1 \bar{w} \, d\eta - \bar{v}' \bar{w} \right]$$

$$+ \bar{w}^{iv}_{\xi} \left(\frac{b^2}{d^2} - 1 \right) \sin \theta \cos \theta + \bar{w}'''(2\gamma\xi \cos 2\theta) \left(\frac{b^2}{d^2} - 1 \right)$$

$$- \bar{w}''(2\gamma^2 \xi \sin 2\theta) \left(\frac{b^2}{d^2} - 1 \right) + \bar{v}^{iv}_{\xi} \left(\sin^2 \theta + \frac{b^2}{d^2} \cos^2 \theta \right)$$

$$- \bar{v}'''(2\gamma\xi \sin 2\theta) \left(\frac{b^2}{d^2} - 1 \right) - \bar{v}''(2\gamma^2 \xi \cos 2\theta) \left(\frac{b^2}{d^2} - 1 \right)$$

$$+ \frac{GJ}{m\Omega^2 L^4} \left(\phi'' \bar{w}'' + \phi' \bar{w}''' \right) + \xi \left(\frac{b^2}{d^2} - 1 \right) \Bigg\{ \bar{w}^{iv}_{\phi} \cos 2\theta$$

$$+ 2\bar{w}'''\phi' \cos 2\theta + \bar{w}''\phi'' \cos 2\theta - 4\gamma \bar{w}'''\phi \sin 2\theta - 4\gamma \bar{w}''\phi' \sin 2\theta$$

$$- 4\gamma^2 \bar{w}''\phi \cos 2\theta - \bar{v}^{iv}_{\phi} \sin 2\theta + 2\bar{v}'''\phi' \sin 2\theta + \bar{v}''\phi'' \sin 2\theta$$

$$+ 4\gamma \bar{v}'''\phi \cos 2\theta + 4\gamma \bar{v}''\phi' \cos 2\theta - 4\gamma^2 \bar{v}'' \sin 2\theta \Bigg\} + \left\{ \bar{H}_2 \right\}$$

$$= - \sin \beta_{pc} \cos \beta_{pc} \left(\frac{\gamma I_{\eta\eta}}{A L^2} \right) \cos 2\theta \qquad (11)$$

$$\ddot{\phi} + f_1 \cos^2 \beta_{pc}(\phi \cos 2\theta) + 2f_1 \cos \beta_{pc}\dot{\bar{v}}' \sin\theta\cos\theta + 2\cos\beta_{pc}\dot{\bar{w}}'\left(f_2 \sin^2\theta + f_3\cos^2\theta\right) + f_4\phi^{iv} + \sin\beta_{pc}\cos\beta_{pc}\left\{\phi''\int_\eta^1 \bar{w}\,d\eta - \phi'\bar{w} - \gamma\bar{w}\right\}$$

$$- \cos^2\beta_{pc}(\phi''Q - \phi'S) - 2\cos\beta_{pc}\left(\phi''\int_\eta^1 \dot{\bar{v}}\,d\eta - \phi'\dot{\bar{v}} - \gamma\dot{\bar{v}}\right)$$

$$+ f_5(2\gamma\phi'\phi'' + \gamma^2\phi'') - f_6\phi'' - f_7\phi'' + f_7(\bar{v}'\bar{w}''' + \bar{v}''\bar{w}'')$$

$$+ f_8\cos^2\beta_{pc}\phi'' - 2\gamma f_8 \sin\beta_{pc}\cos\beta_{pc}\bar{w}'' - f_1\sin\beta_{pc}\cos\beta_{pc}\bar{v}'\cos 2\theta$$

$$- f_1\sin\beta_{pc}\cos\beta_{pc}\bar{w}'\sin 2\theta + f_9\bar{v}''\bar{w}''\cos 2\theta + f_9(\bar{w}''^2 - \bar{v}''^2)\sin\theta\cos\theta + \{\bar{H}_3\}$$

$$= - f_1\cos^2\beta_{pc}\sin\theta\cos\theta - \gamma S\cos^2\beta_{pc} \qquad (12)$$

where

$$Q = \bar{R}(1-\eta) + 0.5(1-\eta^2),\quad S = (\bar{R}+\eta),$$

$$f_1 = \frac{k_{m2}^2 - k_{m1}^2}{k_m^2} = \frac{b^2-d^2}{b^2+d^2},\quad f_2 = \frac{k_{m2}^2}{k_m^2} = \frac{b^2}{b^2+d^2},$$

$$f_3 = \frac{k_{m1}^2}{k_m^2} = \frac{d^2}{b^2+d^2},\quad f_4 = \frac{EC_1}{m\Omega^2k_m^2L^4} = \frac{Eb^2d^2}{12\rho\Omega^2L^4(b^2+d^2)};$$

$$f_5 = \frac{EAk_A^4}{m\Omega^2k_m^2L^4} = \frac{E(b^2+d^2)}{12\rho L^4\Omega^2},\quad f_6 = \frac{EB_1\theta_{pt}'^2}{m\Omega^2k_m^2L^2} = \frac{E\gamma^2b^4}{15\rho\Omega^2L^4(b^2+d^2)};$$

$$f_7 = \frac{GJ}{m\Omega^2k_m^2L^2} = \frac{4Gt^2}{\rho\Omega^2L^2(b^2+d^2)},\quad f_8 = \frac{mk_\lambda^4}{mk_m^2L^2} = \frac{b^2t^2}{12L^2(b^2+d^2)},$$

$$f_9 = \frac{E(I_{\xi\xi} - I_{\eta\eta})}{\rho\Omega^2L^2(I_{\xi\xi} + I_{\eta\eta})} = \frac{E(b^2 - d^2)}{\rho\Omega^2L^2(b^2 + d^2)}, \quad f_{10} = \frac{GJ}{m\Omega^2L^2},$$

$$f_{11} = \xi\left(\frac{b^2}{d^2} - 1\right), \quad f_{12} = \frac{EB_1}{\rho A\Omega^2L^6} = \frac{Eb^4}{180\ \rho\Omega^2L^6},$$

$$f_{13} = \frac{EB_1}{mk_m^2\Omega^2L^4} = \frac{Eb^4}{15\ \rho(b^2 + d^2)\Omega^2L^4},$$

$$\xi = \frac{EI_{\eta\eta}}{\rho AL^4\Omega^2}, \quad \theta = \alpha + \gamma\eta, \quad \theta'_{pt} = \gamma,$$

$$\bar{w}' = \frac{d}{d\eta}(\bar{w}) \quad \text{and} \quad \dot{\bar{w}} = \frac{d}{d\tau}(\bar{w}) \tag{13}$$

Further, the quantities $\bar{H}_1$, $\bar{H}_2$, and $\bar{H}_3$ in equations (10) to (12) are defined by the following:

$$\bar{H}_1 = f_{11}(\sin 2\theta)\left\{\frac{1}{4}\left[2\bar{v}'\bar{v}'''\bar{v}''' + \bar{v}'^2\bar{v}^{iv} + \bar{v}^{iv}\bar{w}'^2 + 2\bar{v}'''\bar{w}'\bar{w}'\right]\right.$$

$$\left. - \left[\bar{v}^{iv}\phi^2 + 4\bar{v}'''\phi\phi' + 2\bar{v}''\phi'^2 + 2\bar{v}''\phi\phi''\right]\right\} + \xi\left(\frac{b^2}{d^2}\sin^2\theta + \cos^2\theta\right)$$

$$\times\left\{(\bar{v}^{iv}\bar{v}'\bar{w}' + 3\bar{v}'''\bar{v}'''\bar{w}' + \bar{v}'\bar{v}'''\bar{w}'' + \bar{v}''^2\bar{w}'') + \left(\bar{w}^{iv}\bar{w}'^2 + 4\bar{w}'\bar{w}''\bar{w}''' + \bar{w}''^3\right)\right\}$$

$$+ f_{11}(\cos 2\theta)\left\{\bar{w}^{iv}\phi^2 + 4\bar{w}'''\phi\phi' + 2\bar{w}''\phi'^2 + 2\bar{w}''\phi\phi''\right\} + f_{11}(\gamma\cos 2\theta)$$

$$\times\left\{(\bar{v}'^2\bar{v}''' + \bar{v}'\bar{v}''^2) + (\bar{v}'''\bar{w}'^2 + \bar{v}''\bar{w}'\bar{w}'') - 4(\bar{v}'''\phi^2 + 2\bar{v}''\phi\phi')\right\}$$

$$+ f_{11}(\gamma^2\cos 2\theta)\left\{-4\bar{w}''\phi^2 + 2\bar{v}'\bar{v}''\bar{w}' + 2\bar{w}''\bar{w}'^2\right\} + f_{11}(\gamma\sin 2\theta)$$

$$\times\left\{-4(\bar{w}'''\phi^2 + 2\bar{w}''\phi\phi') + (2\bar{v}'\bar{v}'''\bar{w}' + 2\bar{v}''^2\bar{w}' + \bar{v}'\bar{v}''\bar{w}''')\right.$$

$$\left. + (2\bar{w}'''\bar{w}'^2 + 3\bar{w}'\bar{w}''^2)\right\} + f_{11}(\gamma^2\sin 2\theta)\left\{-\bar{v}'^2\bar{v}'' - \bar{v}''\bar{w}'^2 + 4\bar{v}''\phi^2\right\}$$

$$- f_{12}\left\{\frac{3\gamma}{2}(\bar{v}'''\phi'^2 + 4\bar{v}''\phi'\phi'' + 2\bar{v}'\phi''^2 + 2\bar{v}'\phi'\phi''') + \gamma^2(v'''\phi' + 2v''\phi''\right.$$

$$\left. + \bar{v}'\phi''' - 2\bar{v}''^2\bar{w}'' - 2\bar{v}'\bar{v}'''\bar{w}'' - 4\bar{v}'\bar{v}''\bar{w}''' - \bar{v}'^2\bar{w}^{iv})\right\}$$

$$+ f_{10}\left\{2\bar{v}''^2\bar{w}'' + 2\bar{v}'\bar{v}'''\bar{w}'' + 4\bar{v}'\bar{v}''\bar{w}''' + \bar{v}'^2\bar{w}^{iv}\right\}$$

$$+ \frac{\cos^2\beta_{pc}}{2}\left\{\bar{w}''\int_\eta^1\int_0^{\bar{\eta}}(\bar{v}'^2 + \bar{w}'^2)\,d\bar{\eta}\,d\eta - \bar{w}'\int_0^\eta(\bar{v}'^2 + \bar{w}'^2)\,d\eta\right\} \tag{14}$$

$$\bar{H}_2 = -f_{11}(\cos 2\theta)\left\{\bar{v}^{iv}\phi^2 + 4\bar{v}'''\phi\phi' + 2\bar{v}''\phi'^2 + 2\bar{v}''\phi\phi''\right\}$$

$$+ f_{11}(\sin 2\theta)\left\{\frac{1}{4}(\bar{w}'^2\bar{w}^{iv} + 6\bar{w}'\bar{w}''\bar{w}''' + 2\bar{w}''^3) + \left(2\bar{v}''\bar{v}'''\bar{w}' + \frac{3}{2}\bar{v}''^2\bar{w}''\right.\right.$$

$$\left.\left. + 2\bar{v}'\bar{v}'''\bar{w}'' + \frac{3}{2}\bar{v}'\bar{v}''\bar{w}''' + \bar{v}'\bar{v}^{iv}\bar{w}' + \frac{\bar{v}'^2\bar{w}^{iv}}{4}\right) - (\bar{w}^{iv}\phi^2 + 4\bar{w}'''\phi\phi' + 2\bar{w}''\phi'^2\right.$$

$$\left. + 2\bar{w}''\phi\phi'')\right\} + \xi\left(\frac{b^2}{d^2}\cos^2\theta\right)+\left\{\sin^2\theta \quad \bar{v}''^3 + 4\bar{v}'\bar{v}''\bar{v}''' + \bar{v}'^2\bar{v}^{iv}\right\}$$

$$+ \xi\left(\frac{b^2}{d^2}\sin^2\theta + \cos^2\theta\right)\left\{\bar{v}''\bar{w}''^2 + 3\bar{v}'\bar{w}''\bar{w}''' + \bar{v}''\bar{w}'\bar{w}''' + \bar{v}'\bar{w}'\bar{w}^{iv}\right\}$$

$$+ f_{11}(\gamma\cos 2\theta)\left\{3\bar{v}''^2\bar{w}' + 4\bar{v}'\bar{v}'''\bar{w}' + 5\bar{v}'\bar{v}''\bar{w}'' + \bar{v}'^2\bar{w}''' - 4(\bar{w}'''\phi^2 + 2\bar{w}''\phi\phi')\right.$$

$$\left. + (\bar{w}'^2\bar{w}''' + 2\bar{w}'\bar{w}''^2)\right\} + f_{11}(\gamma^2\cos 2\theta)\left\{4\bar{v}''\phi^2 - 2\bar{v}''\bar{v}'^2 + 2\bar{v}'\bar{w}'\bar{w}''\right\}$$

$$+ f_{11}(\gamma\sin 2\theta)\left\{4\bar{v}'''\phi^2 + 8\bar{v}''\phi\phi' - 3\bar{v}'\bar{v}''^2 - 2\bar{v}'^2\bar{v}''' + 2\bar{v}'\bar{w}''^2 + 2\bar{v}'\bar{w}'\bar{w}'''\right.$$

$$\left. + \bar{v}''\bar{w}'\bar{w}''\right\} + f_{11}(\gamma^2\sin 2\theta)\left\{-4\bar{v}'\bar{v}''\bar{w}' + 4\bar{w}''\phi^2 - \bar{w}''\bar{v}'^2 - \bar{w}''\bar{w}'^2\right\}$$

$$+ f_{12}\left\{(\gamma^2)(\bar{w}'''\phi' + \bar{w}''\phi'' - \bar{v}''\bar{w}''^2 - 2\bar{v}'\bar{w}''\bar{w}''') + \frac{3\gamma}{2}(\bar{w}'''\phi'^2 + 2\bar{w}''\phi'\phi'')\right\}$$

$$- f_{10}\left\{\bar{v}''\bar{w}''^2 + 2\bar{v}'\bar{w}''\bar{w}'''\right\} + \frac{\cos^2\beta_{pc}}{2}\left\{\bar{v}''\int_\eta^1\int_0^{\bar{\eta}}(\bar{v}'^2 + \bar{w}'^2)\,d\bar{\eta}\,d\eta\right.$$

$$\left. - \bar{v}'\int_0^\eta(\bar{v}'^2 + \bar{w}'^2)\,d\eta\right\} \tag{15}$$

$$\bar{H}_3 = f_9\left\{\phi(\bar{w}''^2 - \bar{v}''^2)\cos 2\theta - 2\phi\bar{v}''\bar{w}''\sin 2\theta\right\}$$

$$- f_{13}\left\{\frac{3}{2}\phi'\phi'' + 3\gamma\phi'\phi'' - 3\gamma(\bar{v}''\bar{w}''\phi' + \bar{v}'\bar{w}'''\phi' + \bar{v}'\bar{w}''\phi'')\right.$$

$$\left. - \gamma^2(\bar{v}''\bar{w}'' + \bar{v}'\bar{w}''')\right\} \tag{16}$$

It may be noted here that the last integral quantities in equations (14) and (15) arise due to the consideration of the third degree terms arising from foreshortening, u_f, in the tension coupling terms $(Tw')'$ and $(Tv')'$ in the flapwise and edgewise equations, respectively. The terms that are premultiplied by f_{10} are the third degree nonlinear terms resulting in the flapwise

and edgewise bending equations respectively from consideration of shearing strains in the strain energy variation. Finally, the terms that are premultiplied by f_{12} in the two bending equations, and those premultiplied by f_{13} in the torsion equation are the nonlinear terms associated with the sectional constant, EB_1. The rest of the terms in equations (14) to (16) are the nonlinear structural terms of third degree, representing the kinematic pitch coupling in the bending and torsion equations (Mil's terms).

The coupled flapwise bending, edgewise bending and torsion equations are solved by the Galerkin method by assuming the dimensionless deflections in the following forms:

$$\bar{w} = \sum_j (w_{oj} + \Delta w_j)\, \psi_j \tag{17}$$

$$\bar{v} = \sum_j (v_{oj} + \Delta v_j)\, \psi_j \tag{18}$$

$$\phi = \sum_j (\phi_{oj} + \Delta \phi_j)\, \theta_j \tag{19}$$

Where w_{oj}, v_{oj}, and ϕ_{oj} are the equilibrium quantities, and Δw_j, Δv_j, and $\Delta \phi_j$ are the perturbation quantities in the generalized coordinates. ψ_j and θ_j are the nonrotating normal mode shape functions for a cantilever beam (ref. 12). It may be noted here that sinusoidal mode shape assumed for the torsional degree of freedom is not compatible with the boundary conditions when warping is included. However, the effect of warping is not significant for large aspect ratio blades considered in this work. Thus, the mode shapes assumed here should produce satisfactory results (ref. 10). Proceeding as in reference 10, one can apply the Galerkin process for the solution of the nonlinear steady state equations and the linearized perturbation equations. Although it is not intended to write all these equations in this work, appendix B contains the additional nonlinear terms in the equilibrium equations that are over and above those presented in reference 10. These additional nonlinear terms in the equilibrium equations can be used for writing the corresponding perturbation equations.

RESULTS AND DISCUSSION

The nonlinear steady state equilibrium equations, and the linearized perturbation equations were solved by using computer programs developed in FORTRAN language. Integrations are performed on the computer using a Gaussian quadrature formula. The programs were run on CRAY/XMP computer at NASA Lewis. Results were also generated from the finite element code, MSC/NASTRAN, using 500 CQUAD4 elements, for various cases of rotating, pretwisted, preconed blades of various thickness ratios. All these results are presented below.

Convergence

The convergence of solutions produced by the Galerkin method with various number of nonrotating normal modes in the assumed solution are presented in table I. For the purpose of comparison of the present theoretical results

obtained from the third degree equations, included in this table are the results from the second degree equations from reference 10, and those obtained from MSC/NASTRAN. In MSC/NASTRAN calculations 250 or 500 CQUAD4 elements were used to model the blade. Furthermore, the centrifugal softening effects were incorporated through suitable DMAP/ALTER procedures, solution 64 which accounts for differential stiffness effects was used to determine the steady state deflections, and subsequently, solution 63 was used to determine the normal modes and frequencies.

From the convergence pattern of the frequency ratios presented in table I, it can be seen that a five-mode Galerkin solution produces the lowest five coupled mode frequencies that show reasonable agreement with MSC/NASTRAN calculations, and that the accuracy of higher mode frequencies can be increased by increasing the number of nonrotating normal modes in the assumed solutions. Furthermore, a comparison of the results from the third degree equations to those from second degree equations indicates that the accuracy of the third and fourth mode coupled frequencies, (corresponding to predominantly first mode torsional and third mode flatwise bending frequencies, respectively, that are closely coupled for this particular blade configuration) is improved to a considerable extent when the present third degree equations are used.

The convergence pattern of the steady state tip deflections produced by the present beam theory is shown in table II. The agreement of present theoretical results to the corresponding ones from MSC/NASTRAN is close here also.

Since a five mode solution ($n = 5$) is found to produce the lowest five coupled mode frequencies and steady state deflections that are in reasonable agreement with the corresponding finite element calculations, further results are generated by using a five-mode Galerkin method solution. Such results are presented and discussed in what follows.

Comparison With Experimental Results

In order to confirm the accuracy of the present theoretical formulation including third degree geometric nonlinearities, results pertaining to typical untwisted and preconed blade cases of thin rotating blades (corresponding to the experimental results reported in ref. 10) are generated. These results are presented in table III together with those obtained from the solution of second degree geometric nonlinear equations and also those from experiment. It should be mentioned here that the elastic modulii used in the theoretical calculations are calibrated values (see ref. 10). A mutual comparison of the three sets of results presented in table III indicates that the results produced by the present, third degree, beam equations are in closer agreement with experimental results than those given by the second degree equations in all cases. Considering the torsional mode frequency (mode 2), it can be seen that the frequency produced by second degree equations for a 22.5° preconed blade with 60° setting angle, and rotating at 3600 rpm, is 489.9 Hz. Corresponding frequency given by the present third degree equations is 563.8 Hz, while the experimental value is 561 Hz. This example illustrates the nature of the third degree geometrically nonlinear elastic terms that are included in the present work. In conclusion, one can state that the results produced by the present set of nonlinear equations are in closer agreement with experimental results than those produced by the second degree equations. Furthermore, a comparison of the results in table III for the particular case of 90° setting angle (blade

chord at root section parallel to the axis of rotation) shows that the effect of geometric nonlinearities for this case of setting angle is not significant, confirming the conclusions drawn in references 9 and 10.

Vibration and Stability of Pretwisted, Preconed, Rotating Blades

In order to ascertain the influence of second and third degree geometric nonlinear elastic terms on the coupled frequencies and steady state deflections, parametric studies are conducted for rotating, pretwisted, preconed blades of various thickness ratio. A typical set of such results are presented in table IV(a) for cases of thin blades, and in table IV(b) for cases of thick blades. The variation of frequency ratio with the rotational speed is shown in figure 2 for a 15° preconed, thin, blade.

From an examination of the results presented in table IV(a), it can be seen that the spurious instabilities observed for the case of thin, preconed blades, when only the second degree geometric nonlinearities are included in the equations, are absent when the third degree geometrically nonlinear elastic terms are included to the second degree equations. Furthermore, the quality of results is more accurate when the present third degree equations are used than is the case with those obtained from the second degree equations, even for the stable configurations in both sets of equations.

In order to acquire a further insight into the nature of the third degree geometric nonlinear terms in coupling the modes, one particular case of rotating, untwisted, thin blade with 15° precone is considered. The variation of the frequency ratios of the lowest five coupled modes with a variation of the rotational parameter, Ω/ω_1, is presented in graphical form in figure 2. The curves representing the frequency ratio variations corresponding to predominantly flapwise bending mode frequencies are marked F_1, F_2 and F_3; and those corresponding to the first torsion mode and the first edgewise bending mode are marked T_1 and S_1 respectively. A second subscript is used to identify the degree of nonlinearity retained in the equations that produced the corresponding results. Thus, the subscript S denotes the second degree equations and the subscript t denotes third degree equations. From an examination of the results presented in figure 2, it can be seen that the flapwise mode frequencies produced by the second, and the third degree equations are quite close (the differences are not apparent in the graphical representation). For this particular case of thin, untwisted blade, a clear-cut coupling trend of the first torsional and first edgewise modes can be seen in figure 2. When a further comparison of these results is made to those produced by MSC/NASTRAN (shown in figure 2 for discrete values of Ω/ω_1, by the symbol Δ), it is observed that the third degree equations produce correct coupling trends, and eliminate the torsional instability that is shown by the second degree equations (refer to the curves marked T_{1S} and T_{1t}). The edgewise mode frequency predicted by the present third degree equations (S_{1t}) is closer to MSC/NASTRAN results than that predicted by the second degree equations (S_{1S}).

An examination of the results presented in table IV(b) corresponding to blades of high thickness ratio indicates that the second degree equations have produced the coupled frequencies that are stable for wide ranges of precones, pretwists, and rotational speeds, unlike those in the case of thin blades. However, the accuracy of results obtained by using the present third degree equations is seen to be much superior. It is interesting to note that while

reporting the results in reference 10, it was believed that the instability predicted by MSC/NASTRAN for the case of the thick blade (d/b = 0.2, γ = 30°, β_{p_c} = 45°, Ω/ω_1 = 1.0) was a true static instability since the second degree equations also predicted an instability at a similar blade rotational speed. However, while the present third degree equations are used, it is found that there is no such instability at this rotational speed. This observation led the authors to re-examine the results of MSC/NASTRAN, and re-run the code with different values for the inplane stiffening parameters assigned to the inplane nodes of CQUAD4 elements for eliminating the inplane rotations. It was found that the spurious instability predicted by the solution sequence 64 could be eliminated by an appropriate selection of the in-plane stiffening parameter. The converged set of results from the present beam theory and MSC/NASTRAN are thus reported in table IV (b) with an appropriate note being made at the bottom of this table. In summary, while MSC/NASTRAN helped in identifying the limitations of second degree geometric nonlinear equations, the third degree geometric nonlinear equations helped in explaining the spurious instabilities, predicted by MSC/NASTRAN in certain cases.

From the mutual comparison of the results produced by the second degree equations, present third degree equations and the corresponding ones from MSC/NASTRAN, presented in tables IV(a) and (b), it can be concluded that the spurious instabilities observed in the case of thin blades while using the second degree equations can be removed by incorporating the third degree elastic terms into the equations. Furthermore, the quality and reliability of the results produced by the present third degree equations are much superior to those obtained from second degree equations even in the case of thick blades. For configurations that are more complex than those considered in this work, it may however be necessary to include further nonlinear terms, since it appears from the present investigation (refer tables III and IV), that the second degree elastic terms have a softening effect (or stiffening effect depending on the setting angle and precone) on the torsional modes while the third degree terms have a stiffening (or softening) effect on the torsional modes (thus eliminating the spurious torsional instabilities as shown in table IV(a)). While it can be contended from the present investigation that the range of applicability of the nonlinear equations is extended to practical rotor blade operating conditions by the inclusion of the third degree elastic terms, these are by no means, the ultimate form of the equations. Further studies are therefore necessary for making the beam theory equations applicable for more extreme blade configurations.

Finally, it is believed useful to determine the relative importance of the various nonlinear elastic terms of third degree that enter into the equations from the consideration of various primary sources in the variational formulation, such as the normal strain or shearing strains, coupling between tension and bending or torsion etc. Such a study will help in identifying the more important nonlinear terms and in eliminating the relatively less important terms, depending on the type of problem being addressed. To accomplish this objective, results are produced by solving the present beam theory equations by successively discarding one key effect each time, for a few, typical, preconed blade configurations. These results are shown in table V, together with corresponding results from MSC/NASTRAN. Thus, the results presented under the column marked case (1) are obtained from the solution of the third degree nonlinear equations shown by equations (10) to (12). The results presented under the column marked case (2) are obtained by ignoring the third degree nonlinear

terms arising from the tension terms (Tw')' and (Tv')' in the two bending equations. It may be noted here, by inspecting equation (4), that these third degree elastic terms result from the consideration of foreshortening due to bending, u_F, in the tension expression. Results shown under the column marked case (3) are obtained by further neglecting the third degree nonlinear terms that arise from the shearing strains. These terms can be identified in equations (A24) and (A25) as those which are associated with GJ. Next, by inspecting equations (A24) to (A26) of appendix A, one can find a group of terms in each of these equations premultiplied by the sectional constant, EB_1. These terms were addressed in reference 15, and subsequently in reference 16, for specialized cases of torsional motion completely uncoupled from bending motions of pretwisted blades. It was established that, under the conditions of completely uncoupled torsional motion, the nonlinear terms containing the torsional deformation ϕ and associated with EB_1, are very important. However, their effect in the presence of the kinematic pitch coupling terms is not completely understood. In order to verify the importance of the nonlinear terms associated with the sectional constant EB_1 in the presence of Mil's terms, results are first produced by ignoring the nonlinear EB_1 terms in the two bending equations together with those already discarded in case (3). These are presented under the column marked case (4). The results obtained by ignoring all the nonlinear terms associated with EB_1 in the torsion equation, over and above those discarded in the previous case, are presented under the column, case (5). Finally, the results obtained by further discarding Mil's terms of third degree in the two bending equations but retaining them in the torsion equation, are presented under the column marked case (6). The last column of table V shows the results produced by the second degree equations addressed in reference 10.

From an examination of these results presented in table V, one can observe that for the blade configurations addressed in this work, the influence of third degree elastic terms arising from the foreshortening due to bending, u_F, in the bending equations is of the order of one to two percent on the steady state deflections and frequencies. These third degree terms are seen to produce a softening effect on the lowest four coupled frequencies. Since the corresponding nonlinear terms are not included in the definition of the tension, given in equation (A28) and used for eliminating u', in the torsion equation, the exact effect of all these terms is not known. However, in view of the close agreement of theoretical results and NASTRAN generated results, (compare results of MSC/NASTRAN to corresponding ones under cases (1) and (2)), it is believed that the third degree terms arising from tension coupling can safely be ignored in all the equations consistently. Further, from a comparison of results presented under cases (3) to (5) with the corresponding ones from case (2) and MSC/NASTRAN, it is seen that the third degree nonlinear terms arising from shearing strains, and also the nonlinear terms associated with the sectional constant, EB_1, produce insignificant changes on the steady-state deflections and on the coupled frequencies. Thus, these nonlinear terms can be discarded for blade configurations addressed in this work. Next, the influence of discarding third degree Mil's terms in the bending equations is seen to be quite significant on the steady state deflections and frequencies of pretwisted blades. However, the results pertaining to untwisted blades obtained by discarding third degree Mil's terms in bending equations but retaining in the torsion equation are not affected, and remain identical to those obtained from consideration of these terms in all equations. The reason for this would become clear by inspecting the steady state deflections for this particular case. It can be seen that for this particular case of untwisted blade with

$\gamma = \alpha = 0°$, the only significant steady state deflection is $\bar{w}$. This gives rise to significant equilibrium coordinates w_{0k} and trivial values for v_{0k} and ϕ_{0k}. Furthermore, since $\alpha = \gamma = 0$, most of the terms in the equations (A24) to (A26) vanish, and the majority of the still remaining terms of third degree will vanish in the perturbation equations due to multiplication of w_{0k} with v_{0k} or ϕ_{0k}. However, there appears to be a need for caution in discarding Mil's terms, since, in the presence of aerodynamic forces (which are not addressed in this work), the equilibrium coordinates v_{0k} and ϕ_{0k} may not be trivial. In these or similar circumstances, it would be desirable to retain third degree Mil's terms in all the equations, whether or not the blade is pretwisted.

After identifying the necessity and the importance of the third degree elastic terms as discussed above, there remain certain questions concerning the rationale of the ordering scheme in discarding the inertial terms. As stated in the introduction to this work,and also in appendix A, several nonlinear inertial terms were discarded in reference 10 based upon the ordering scheme followed therein. For consistency of retaining terms when the third degree elastic terms are added to the second degree equations of reference 10, one has to consider the influence of the several inertial terms that were discarded earlier, before any attempts are made to discard these terms in the presence of the third degree elastic terms. In order to answer these questions, at least partially, several inertial terms that were discarded previously of order $O(\epsilon^4)$, and arising due to rotational effects (which are not involved with any time derivatives), are added to the present torsion equation. On solving the resulting coupled equations, it is found that the changes in the frequencies were at the fifth significant figure level for a 15° preconed, 30° pretwisted, rotating, thin blade ($d/b = 0.05$, $\Omega/\omega_1 = 1.0$, $\alpha = 0°$). In view of this observation, it appears rational to discard the large number of higher order inertial terms without losing any appreciable accuracy on the final results.

CONCLUDING REMARKS

The coupled flapwise bending, edgewise bending and torsion equations of dynamic motion of rotating, linearly pretwisted, and large preconed blades of symmetric cross section including third degree elastic, geometrically nonlinear terms are derived. These equations are solved by using the Galerkin method and a linear perturbation procedure. Parametric results are generated to ascertain the necessity and the influence of third degree elastic terms. Comparisons of present theoretical results are made to those produced by the finite element code, MSC/NASTRAN, allowing for geometric nonlinear effects; and to those produced from an in-house experimental test rig. Close agreement of the present theoretical results to the corresponding results from other methods is observed for the parametric range studied. The following specific conclusions have emerged in the course of this investigation:

(1) For the dynamic analysis of large preconed blades, the second degree geometric nonlinear equations are not adequate. The validity of beam theory equations can be extended to cover large precones and practical rotational speeds by adding the third degree elastic terms involving the kinematic pitch coupling to the existing second degree equations. It is found that third degree Mil's terms counteract the influence of the corresponding second degree terms, and thus, the spurious instabilities that one might observe in the case

of thin, rotating blades while using the second degree equations can be suppressed by considering third degree Mil's terms. It can thus be concluded that inclusion of at least third degree Mil's terms into the equations is absolutely necessary.

(2) For the blade configurations addressed in this work, the third degree nonlinear elastic terms resulting from the normal strain are found to be important. The third degree nonlinear terms resulting from the shearing strains produce insignificant changes on the coupled frequencies. Thus, the nonlinear terms of third degree that result from the variation of shearing strains may be neglected in the equations of motion. Furthermore, the nonlinear terms associated with the sectional constant, EB_1, are also found to produce insignificant changes in the steady state deflections and coupled frequencies. In the presence of the third degree Mil's terms in the equations, these nonlinear terms may also be discarded.

(3) The present approximation used for defining the tension (only in terms of linear parameters in the equation for T) appears to be adequate. Thus, defining the tension in all the equations consistently in terms of linear variables does not result in any great loss of accuracy in the final results.

(4) The influence of higher order inertial terms, other than those retained in the present equations, appears to be insignificant on the final results.

APPENDIX A - NONLINEAR CURVATURE EXPRESSIONS AND EQUATIONS OF MOTION

In order to derive geometric nonlinear equations, it is necessary to differentiate between the deformed and undeformed configurations, and to include nonlinear terms in the strain displacement relations. If nonlinear terms up to third degree are to be considred in the equations, one has to derive the curvature expressions having terms up through third degree in the deformation variables and also the third degree nonlinear transformation matrix between the deformed and undeformed blade-fixed coordinates. In arriving at the deformed configuration, one starts with the undeformed configuration and imposes a sequence of rotional transformations on the blade-fixed coordinates. Thus, deformations u, v, w and ϕ displace the undeformed blade-fixed coordinate system $x_0y_0z_0$ to xyz (fig. 3), and rotate xyz to $x_3y_3z_3$ where x_3 is tangent to the deformed elastic axis. The rotation of the triad xyz to the final position $x_3y_3z_3$ may be expressed in terms of the Eulerian type angles. Of the several possible rotational transformation sequences, the one employed here is the flap-lag-pitch rotational transformation sequence. Further details on the rotational transformation sequences can be found in reference 13. Thus, following an approach identical to that given in reference 13, and retaining terms up through third degree in u, v, w and ϕ, the nonlinear curvature expressions are derived for a twisted rotor blade undergoing transverse bending in two planes (flapwise along the z-zxis, edgewise along the y-axis), extension (u) and torsion (ϕ). In what follows, we present the final equations dispensing with all the intermediate and lengthy algebra.

When no assumptions are made on the magnitude of deformation the transformation matrix between the deformed and undeformed blade fixed coordinates is given by the following equation if nonlinear terms up through third degree are retained:

$$\begin{Bmatrix} x_3 \\ y_3 \\ z_3 \end{Bmatrix} = \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{A1}$$

where

$$\ell_1 = 1 - \frac{v'^2 + w'^2}{2} + u'(v'^2 + w'^2) \tag{A2}$$

$$m_1 = v' - u'v' - \frac{v'}{2}(v'^2 + w'^2 - 2u') \tag{A3}$$

$$n_1 = w' - u'w' - \frac{w'}{2}(v'^2 + w'^2 - 2u'^2) \tag{A4}$$

$$\ell_2 = -\cos\theta_{pt}\left\{v' - u'v' + \phi w' - u'w'\phi - \frac{v'\phi^2}{2} + v'\left(u'^2 - \frac{v'^2}{2} - w'^2\right)\right\}$$
$$- \sin\theta_{pt}\left\{w' - u'w' - \phi v' + u'v'\phi - \frac{w'\phi^2}{2} + w'\left(u'^2 - \frac{w'^2}{2}\right)\right\} \tag{A5}$$

$$m_2 = \cos\theta_{pt}\left\{1 - \frac{v'^2}{2} - \frac{\phi^2}{2} + u'v'^2\right\} - \sin\theta_{pt}\left\{\phi - \frac{v'^2\phi}{2} - \frac{\phi^3}{6}\right\} \qquad (A6)$$

$$n_2 = \cos\theta_{pt}\left\{\phi - v'w' + 2u'v'w' - \frac{w'^2\phi}{2} - \frac{\phi^3}{6}\right\}$$

$$+ \sin\theta_{pt}\left\{1 - \frac{w'^2}{2} - \frac{\phi^2}{2} + v'w'\phi + u'w'^2\right\} \qquad (A7)$$

$$\ell_3 = \cos\theta_{pt}\left\{-w' + v'\phi + u'w' - u'v'\phi - w'u'^2 + \frac{w'^3}{2} + \frac{w'\phi^2}{2}\right\}$$

$$+ \sin\theta_{pt}\left\{v' - u'v' + w'\phi - u'w'\phi + v'u'^2 - \frac{v'^3}{3} - v'w'^2 - \frac{v'\phi^2}{2}\right\} \qquad (A8)$$

$$m_3 = -\cos\theta_{pt}\left\{\phi - \frac{v'^2\phi}{2} - \frac{\phi^3}{6}\right\} - \sin\theta_{pt}\left\{1 - \frac{v'^2}{2} + u'v'^2 - \frac{\phi^2}{2}\right\} \qquad (A9)$$

$$n_3 = \cos\theta_{pt}\left\{1 - \frac{w'^2}{2} - \frac{\phi^2}{2} + u'w'^2 + v'w'\phi\right\}$$

$$+ \sin\theta_{pt}\left\{-\phi + v'w' + \frac{w'^2\phi}{2} + \frac{\phi^3}{6} - 2u'v'w'\right\} \qquad (A10)$$

The direction cosines given by equations (2) to (10) obey the orthogonality and normality conditions up to third degree terms.

If assumptions corresponding to the case of small deformations I discussed in reference 13 are imposed, and terms up through third degree are retained, one obtains the transformation matrix between the deformed and undeformed coordinate systems as follows:

$$\begin{Bmatrix} x_3 \\ y_3 \\ z_3 \end{Bmatrix} = \left[\begin{array}{c|c|c} 1 - \frac{v'^2 + w'^2}{2} & v' & w' \\ \hline \cos\theta_{pt}\left\{-v' - \phi w' + \frac{v'}{2}(\phi^2 + w'^2)\right\} & \cos\theta_{pt}\left\{1 - \frac{v'^2}{2} - \frac{\phi^2}{2}\right\} & \cos\theta_{pt}\left\{\phi - v'w' - \frac{\phi w'^2}{2} - \frac{\phi^3}{6}\right\} \\ + \sin\theta_{pt}\left\{-w' + \phi v' + \frac{w'}{2}(\phi^2 - v'^2)\right\} & + \sin\theta_{pt}\left\{-\phi + \frac{v'^2\phi}{2} + \frac{\phi^3}{6}\right\} & + \sin\theta_{pt}\left\{1 - \frac{w'^2}{2} - \frac{\phi^2}{2} + v'w'\phi\right\} \\ \hline \cos\theta_{pt}\left\{-w' + v'\phi + \frac{w'\phi^2}{2} - \frac{w'v'^2}{2}\right\} & \cos\theta_{pt}\left\{-\phi + \frac{\phi v'^2}{2} + \frac{\phi^3}{6}\right\} & \cos\theta_{pt}\left\{1 - \frac{w'^2}{2} - \frac{\phi^2}{2} + v'w'\phi\right\} \\ + \sin\theta_{pt}\left\{v' + w'\phi - \frac{v'w'^2}{2} - \frac{v'\phi^2}{2}\right\} & + \sin\theta_{pt}\left\{-1 + \frac{v'^2}{2} + \frac{\phi^2}{2}\right\} & + \sin\theta_{pt}\left\{-\phi + v'w' + \frac{\phi w'^2}{2} + \frac{\phi^3}{6}\right\} \end{array}\right] \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

In obtaining equation (11), it is assumed that the elongations and shears are negligible compared to unity, and the squares of derivative of the extensional deformation of the elastic axis is negligible compared to the square of bending slopes. The direction cosines given in equation (11) obey the normality and orthogonality conditions to third degree terms. It may also be noted that the torsion variable, θ, is the sum of pretwist and elastic twist of the blade at any section. Thus, for a linearly pretwisted blade with total pretwist over the length L of the blade, θ at any distance x from the root section is given by

$$\theta = \alpha + \gamma\eta + \phi = \theta_{pt} + \phi \tag{A12}$$

where α is the blade setting angle and $\eta = x/L$.

The bending curvatures ω_{y3} and ω_{z3}, and the torsional curvature ω_{x3} can be obtained by using either the direction cosines given earlier, or by using the projections of the space derivatives of the rotation angles with respect to the deformed coordinate S_3 along the elastic axis of blade. For the case of small deformation level I (ref. 13), the resulting curvature expressions with terms up through third degree in u,v,w, and ϕ, and for flap-lag-pitch transformation sequence of rotations are:

$$\omega_{x3} = \phi'_{pt} + \phi' - v'w'' \tag{A13}$$

$$\omega_{y3} = \cos\theta_{pt}\left\{-w'' + \phi v'' - v'v''w' - \frac{w''w'^2}{2} + \frac{w''\phi^2}{2}\right\} + \sin\theta_{pt}\left\{v'' + w''\phi + \frac{v''v'^2}{2} - \frac{v''\phi^2}{2}\right\} \tag{A14}$$

$$\omega_{z3} = \cos\theta_{pt}\left\{v'' + w''\phi + \frac{v''v'^2}{2} - \frac{v''\phi^2}{2}\right\} + \sin\theta_{pt}\left\{w'' - v''\phi + v'v''w' + \frac{w''w'^2}{2} - \frac{w''\phi^2}{2}\right\} \tag{A15}$$

If $\bar{r}_0$ and $\bar{r}_1$ are the undeformed and deformed position vectors of an arbitrary point on the blade cross section, then the Green's strain tensor, ε_{ij}, can be obtained from the following equation:

$$d\bar{r}_1 \cdot d\bar{r}_1 - d\bar{r}_0 \cdot d\bar{r}_0 = 2\left[ds\ d\hat{\eta}\ d\hat{\xi}_j\right]\left[\varepsilon_{ij}\right]\begin{Bmatrix} ds \\ d\hat{\eta} \\ d\hat{\xi} \end{Bmatrix} \tag{A16}$$

Proceeding on the same lines as presented in reference 14, one can obtain the normal strain γ_{xx} and the shearing strains $\gamma_{x\eta}$ and $\gamma_{x\xi}$. In terms of the curvature expressions, the normal strain γ_{xx}, and shearing strains $\gamma_{x\xi}$ and $\gamma_{x\eta}$ can be shown as

$$\gamma_{xx} = \left\{\varepsilon - \lambda\phi'' - \hat{\eta}\omega_{z3} + \hat{\xi}\omega_{y3} + \frac{\hat{\eta}^2}{2}\omega_{z3}^2 + \frac{\hat{\xi}^2}{2}\omega_{y3}^2 - \hat{\eta}\hat{\xi}\,\omega_{y3}\omega_{z3} + \frac{\hat{\eta}^2 + \hat{\xi}^2}{2}\omega_{x3}^2\right\} \tag{A17}$$

$$\gamma_{x\eta} = -(\hat{\xi} + \lambda_\eta)\omega_{x3} \tag{A18}$$

$$\gamma_{x\xi} = -(\hat{\eta} - \lambda_\xi)\omega_{x3} \tag{A19}$$

where

$$\varepsilon = u' + \frac{v'^2 + w'^2}{2} \tag{A20}$$

Considering foreshortening due to bending explicitly as in reference 3, the expression for the normal strain to be used in the development of strain energy can be shown to be the following:

$$\begin{aligned}\gamma_{xx} = {} & u' - \lambda\phi'' - y_o\left\{v'' + w''\phi + \frac{v''v'^2}{2} - \frac{v''\phi^2}{2}\right\} \\ & - z_o\left\{w'' - v''\phi + v'w'v'' + \frac{w''w'^2}{2} - \frac{w''\phi^2}{2}\right\} \\ & + \frac{y_o^2}{2}\left\{v''^2 + 2v''w''\phi\right\} + \frac{z_o^2}{2}\left\{w''^2 - 2v''w''\phi\right\} \\ & + y_o z_o\left\{v''w'' - v''^2\phi + w''^2\phi\right\} + \frac{\hat{\eta}^2 + \hat{\xi}^2}{2}\left\{\phi'^2 + 2\theta'_{pt}\phi'\right. \\ & \left. - 2v'w''\phi' - 2\theta'_{pt}\,v'w''\right\}\end{aligned} \tag{A21}$$

where

$$y_o = (\hat{\eta}\cos\theta_{pt} - \hat{\xi}\sin\theta_{pt}) \tag{A22}$$

$$z_o = (\hat{\eta}\sin\theta_{pt} + \hat{\xi}\cos\theta_{pt}) \tag{A23}$$

Since it is of interest in the current investigation to determine the additional third degree elastic terms only, we present here only such terms as are obtained from the variation of γ_{xx}, $\gamma_{x\xi}$, and $\gamma_{x\eta}$ in the strain energy variation. Furthermore, we assume the blade to possess symmetry about the two principal axes. For this specialized case shown in figure 2, the additional nonlinear terms of third degree, over and above those derived in reference 10, are presented below for the flatwise bending (H_1), edgewise bending (H_2), and torsion (H_3) equations, respectively.

Additional elastic terms in flapwise bending equation:

$$H_1 = \Big[(EI_{\xi\xi} - EI_{\eta\eta})\Big\{-v''\phi^2 \sin 2\theta_{pt} + \frac{v''}{2}(v'^2 + w'^2)\sin\theta_{pt}\cos\theta_{pt} + w''\phi^2\cos 2\theta_{pt}\Big\} + \Big(EI_{\xi\xi}\sin^2\theta_{pt} + EI_{\eta\eta}\cos^2\theta_{pt}\Big)(v'v''w' + w'^2w'') - \frac{EB_1}{2}\Big\{3\theta'_{pt}\,v'\phi'^2 + 2\theta'^2_{pt}(v'\phi' - v'^2w'')\Big\}\Big] + [GJv'^2w'']'' - \Big[(EI_{\xi\xi} - EI_{\eta\eta})\sin\theta_{pt}\cos\theta_{pt}\Big\{v'v''^2 + v''w'w''\Big\} + \Big(EI_{\xi\xi}\sin^2\theta_{pt} + EI_{\eta\eta}\cos^2\theta_{pt}\Big)\Big\{v'v''w'' + w'w''^2\Big\}\Big]' - \frac{\cos^2\beta_{pc}}{2}\Big[w''\int_x^L u_F dx - w'u_F\Big] \tag{A24}$$

Edgewise bending:

$$H_2 = \Big[(EI_{\xi\xi} - EI_{\eta\eta})\Big\{-v''\phi^2\cos 2\theta_{pt} + v'v''w'\sin 2\theta_{pt} - w''\phi^2\sin 2\theta_{pt} + \frac{w''}{2}(v'^2 + w'^2)\sin\theta_{pt}\cos\theta_{pt}\Big\} + \Big(EI_{\xi\xi}\cos^2\theta_{pt} + EI_{\eta\eta}\sin^2\theta_{pt}\Big)v'^2v'' + \Big(EI_{\xi\xi}\sin^2\theta_{pt} + EI_{\eta\eta}\cos^2\theta_{pt}\Big)v'w'w''\Big]'' - [GJv'w''^2]' - \Big[(EI_{\xi\xi} - EI_{\eta\eta})\sin\theta_{pt}\cos\theta_{pt}(v'v''w'' + v''^2w') + \Big(EI_{\xi\xi}\cos^2\theta_{pt} + EI_{\eta\eta}\sin^2\theta_{pt}\Big)v'v''^2 + \Big(EI_{\xi\xi}\sin^2\theta_{pt} + EI_{\eta\eta}\cos^2\theta_{pt}\Big)v''w'w'' - \frac{EB_1}{2}\Big\{2\theta'^2_{pt}\,w''\phi' + 3\theta'_{pt}\,w''\phi'^2 - 2\theta'^2_{pt}\,v'w''^2\Big\}\Big]' - \frac{\cos^2\beta_{pc}}{2}\Big[v''\int_x^L u_F dx - v'u_F\Big] \tag{A25}$$

Torsion:

$$H_3 = (EI_{\xi\xi} - EI_{\eta\eta})\Big\{\phi(w''^2 - v''^2)\cos 2\theta_{pt} - 2v''w''\phi\sin 2\theta_{pt}\Big\} - \Big[\frac{EB_1}{2}\Big\{\phi'^3 + 3\theta'_{pt}\phi'^2 + 2\theta'^2_{pt}\phi' - 6\theta'_{pt}v'w''\phi' - 2\theta'^2_{pt}v'w''\Big\}\Big]' \tag{A26}$$

It may be noted that the integral quantities associated with foreshortening, u_F, in equations (A24) and (A25) are the third degree nonlinear terms arising from (Tw')' and (Tv')'. Furthermore, the underlined terms in equation (A26) are not of third degree. While the term $2\theta'_{pt}{}^2\,\phi'$ was considered in the equations of reference 10, the term $3\theta'_{pt}\,\phi'^2$ was omitted based upon the ordering scheme followed therein. These terms are rewritten here for the sake of avoiding any confusion. Furthermore, these torsional terms, together with ϕ'^3, were discussed by Houbolt and Brooks (ref. 15). Furthermore, it should be pointed out here that the effect of higher order terms in the kinetic energy on the frequencies and steady state deflections was shown to be of trivial importance in reference 10. Thus, no attempts are made here to obtain the third degree nonlinear expressions pertaining to the variation of the kinetic energy.

Finally it is of interest to present the contribution of various terms obtained from the variation of the strain energy and pertaining to the extensional equation of motion. From the expression of normal strain given in equation (A21), one can obtain the following elastic terms belonging to the extensional equation of motion:

$$\begin{aligned}
&\Bigg[EA\left\{u' + \frac{k_A^2}{2}\left(\phi'^2 + 2\theta'_{pt}\phi' - 2v'w''\phi' - 2\theta'_{pt}\,v'w''\right)\right\} \\
&\quad + \frac{1}{2}\left(EI_{\xi\xi}\cos^2\theta_{pt} + EI_{\eta\eta}\sin^2\theta_{pt}\right)\left\{v''^2 + 2v''w''\phi\right\} \\
&\quad + \frac{1}{2}\left(EI_{\xi\xi}\sin^2\theta_{pt} + EI_{\eta\eta}\cos^2\theta_{pt}\right)\left\{w''^2 - 2v''w''\phi\right\} \\
&\quad + (EI_{\xi\xi} - EI_{\eta\eta})\sin\theta_{pt}\cos\theta_{pt}\left\{v''w'' - v''^2\phi + w''^2\phi\right\}\Bigg]' \qquad \text{(A27)}
\end{aligned}$$

From the foregoing expression, one can define the tension T on the blade cross section including third degree nonlinearities as follows by the conditions of equilibrium of elastic and inertial forces in the extension equation (refer to eqs. (95) and (96) of ref. 3):

$$\begin{aligned}
T = EA&\left\{\underline{u'} + \frac{k_A^2}{2}\left(\phi'^2 + \underline{2\theta'_{pt}\phi'} - 2v'w''\phi' - 2\theta'_{pt}v'w''\right)\right\} \\
&+ \frac{1}{2}\left(EI_{\xi\xi}\cos^2\theta_{pt} + EI_{\eta\eta}\sin^2\theta_{pt}\right)\left\{v''^2 + 2v''w''\phi\right\} \\
&+ \frac{1}{2}\left(EI_{\xi\xi}\sin^2\theta_{pt} + EI_{\eta\eta}\cos^2\theta_{pt}\right)\left\{w''^2 - 2v''w''\phi\right\} \\
&+ (EI_{\xi\xi} - EI_{\eta\eta})\sin\theta\cos\theta\left\{v''w'' - v''^2\phi + w''^2\phi\right\} \qquad \text{(A28)}
\end{aligned}$$

However, while solving the second degree equations in reference 10, only the underlined terms were retained in equation (A28) to define the tension, and the rest of the second degree geometric nonlinear terms were discarded based upon the ordering scheme employed therein. The resulting equation was used to eliminate u' from the torsion equation. Even with the inclusion of the linear term $EA\, k_A^2\, \theta'_{pt}\Phi'$ in the tension equation, several nonlinear terms result in the torsion equation due to the tension coupling. The effect of all such nonlinear terms was found to be negligible on the steady state deflections and the coupled frequencies. In view of this observation of reference 10, it is believed that all the nonlinear terms in equation (A28) can be safely discarded without affecting the final results to any appreciable extent.

APPENDIX B - ADDITIONAL NONLINEAR TERMS IN THE EQUILIBRIUM EQUATIONS

The second degree modal equations resulting from the Galerkin process of the coupled bending-torsional equations motion of rotating, pretwisted, preconed blades were presented in reference 10. The additional nonlinear terms that arise in the equilibrium equations due to the consideration of the third degree geometric nonlinearities are presented in what follows. These additional nonlinear terms are to be added to the corresponding elements of the matrix representing the equilibrium equations of reference 10. For this purpose, the equilibrium equations are defined in the following matrix form,

$$\left[\begin{bmatrix}\text{Linear and second degree nonlinear terms (ref. 10)}\end{bmatrix} + \left[\begin{array}{c|c|c} F1 & F2 & F3 \\ \hline E1 & E2 & E3 \\ \hline T1 & T2 & T3 \end{array}\right]\right]\begin{Bmatrix} w_{oj} \\ v_{oj} \\ \phi_{oj} \end{Bmatrix} = \{B\} \tag{B1}$$

where F1, F2, .. T3 are the additional nonlinear terms. These elements are defined as follows:

$$\begin{aligned} F1 = {} & f_{11}[w_{ok}v_{o\ell}\,A11_{ijk\ell} + \phi_{ok}\phi_{o\ell}\,A3T_{ijk\ell}] + \xi\,[v_{ok}v_{o\ell}A4T_{ijk\ell} + w_{ok}w_{o\ell}A5T_{ijk\ell}] \\ & + f_{11}\Big[\gamma\, w_{ok}v_{o\ell}C11_{ijk\ell} - 4\gamma^2\,\phi_{ok}\phi_{o\ell}\,C3T_{ijk\ell} + 2\gamma^2\,v_{ok}v_{o\ell}C4T_{ijk\ell} \\ & + 2\gamma^2\,w_{ok}w_{o\ell}C4T_{ijk\ell} - 4\gamma\,\phi_{ok}\phi_{o\ell}C5T_{ijk\ell} + 2\gamma v_{ok}v_{o\ell}C6T_{ijk\ell} + 2\gamma w_{ok}w_{o\ell}C7T_{ijk\ell} \\ & - \gamma^2 w_{ok}v_{o\ell}C8T_{ijk\ell}\Big] + f_{12}\,\gamma^2\,v_{ok}v_{o\ell}D2T_{ijk\ell} + f_{10}v_{ok}v_{o\ell}D21_{ijk\ell} \\ & + \frac{\cos^2\beta_{pc}}{2}\,(v_{ok}v_{o\ell} + w_{ok}w_{o\ell})\,UF_{ijk\ell} \end{aligned} \tag{B2}$$

$$\begin{aligned} F2 = {} & f_{11}[v_{ok}v_{o\ell}A11_{ijk\ell} - \phi_{ok}\phi_{o\ell}A21_{ijk\ell}] + f_{11}\Big[\gamma v_{ok}v_{o\ell}C11_{ijk\ell} \\ & - 4\gamma\,\phi_{ok}\phi_{o\ell}C21_{ijk\ell} - \gamma^2\,v_{ok}v_{o\ell}C81_{ijk\ell} + 4\gamma^2\phi_{ok}\phi_{o\ell}C9T_{ijk\ell}\Big] \\ & - \frac{3\gamma}{2}\,f_{12}\,\phi_{ok}\phi_{o\ell}D11_{ijk\ell} \end{aligned} \tag{B3}$$

$$F3 = -\gamma^2\,f_{12}\,v_{ok}T1_{ijk} \tag{B4}$$

$$E1 = f_{11}\,[W_{ok}W_{o\ell}A7T_{ijk\ell} - \Phi_{ok}\Phi_{o\ell}A2T_{ijk\ell}] + f_{11}\Big[-\gamma^2\,W_{ok}W_{o\ell}C8T_{ijk\ell}$$
$$+ 4\gamma^2\,\Phi_{ok}\Phi_{o\ell}C9T_{ijk\ell} - 4\gamma\Phi_{ok}\Phi_{o\ell}C2T_{ijk\ell} + \gamma\,W_{ok}W_{o\ell}D8T_{ijk\ell}\Big]$$
$$+ \frac{3\gamma}{2}\,f_{12}\Phi_{ok}\Phi_{o\ell}D6T_{ijk\ell} \qquad (B5)$$

$$E2 = f_{11}\Big[4\gamma\,\Phi_{ok}\Phi_{o\ell}C5T_{ijk\ell} - 2\gamma\,V_{ok}V_{o\ell}C7T_{ijk\ell} + 2\gamma\,W_{ok}W_{o\ell}C6T_{ijk\ell}$$
$$- \gamma^2\,V_{ok}W_{o\ell}D3T_{ijk\ell} + \gamma\,V_{ok}W_{o\ell}D4T_{ijk\ell} + 4\gamma^2\Phi_{ok}\Phi_{o\ell}C3T_{ijk\ell}$$
$$- 2\gamma^2\,V_{ok}V_{o\ell}C4T_{ijk\ell} + 2\gamma^2\,W_{ok}W_{o\ell}C4T_{ijk\ell}\Big] - \gamma^2\,f_{12}\,W_{ok}W_{o\ell}D5T_{ijk\ell}$$
$$- f_{10}\,W_{ok}W_{o\ell}D5T_{ijk\ell} + \frac{\cos^2\beta_{pc}}{2}\,(V_{ok}V_{o\ell} + W_{ok}W_{o\ell})\,UF_{ijk\ell}$$
$$+ f_{11}\,[-\Phi_{ok}\Phi_{o\ell}A3T_{ijk\ell} + V_{ok}W_{o\ell}A6T_{ijk\ell}] + \xi\,[V_{ok}V_{o\ell}A8T_{ijk\ell} + W_{ok}W_{o\ell}A9T_{ijk\ell}] \qquad (B6)$$

$$E3 = \gamma^2\,f_{12}\,W_{ok}T4_{ijk} \qquad (B7)$$

$$T1 = 0 \qquad (B8)$$

$$T2 = \gamma^2\,f_{13}\,W_{ok}T8_{ijk} \qquad (B9)$$

$$T3 = f_9\,[W_{ok}W_{o\ell} - V_{ok}V_{o\ell}]\,B1T_{ijk\ell} - 2f_9\,W_{ok}V_{o\ell}B2T_{ijk\ell}$$
$$- \frac{3}{2}\,f_{13}\,\Phi_{ok}T7_{ijk} + 3\gamma\,f_{13}\,W_{ok}V_{o\ell}D7T_{ijk\ell} - 3\gamma\,f_{13}\,\Phi_{ok}T7_{ijk} \qquad (B10)$$

The integral quantities $A1T_{ijk\ell}\ldots D8T_{ijk\ell}$ appearing in the equations shown above are defined below.

$$A1T_{ijk\ell} = \int_0^1 \Big\{\psi_i\;0.5\;\psi_j'\psi_k''\psi_\ell''' + 0.25\;\psi_j'\psi_k'\psi_\ell^{iv}\Big\}\sin 2\theta\;d\eta$$

$$A2T_{ijk\ell} = \int_0^1 \psi_i\Big\{\psi_j^{iv}\theta_k\theta_\ell + 4\psi_j'''\theta_k\theta_\ell' + 2\psi_j''\theta_k'\theta_\ell' + 2\psi_j''\theta_k\theta_\ell''\Big\}\sin 2\theta\;d\eta$$

$$A3T_{ijk\ell} = \int_0^1 \psi_i \left\{ \psi_j^{iv} \theta_k \theta_\ell + 4\psi_j''' \theta_k \theta_\ell' + 2\psi_j'' \theta_k' \theta_\ell' + 2\psi_j'' \theta_k \theta_\ell'' \right\} \cos 2\theta \, d\eta''$$

$$A4T_{ijk\ell} = \int_0^1 \psi_i \left\{ \psi_j \psi_k \psi_\ell^{iv} + 3\psi_j \psi_k''' \psi_\ell' + \psi_j'' \psi_k''' \psi_\ell' + \psi_j'' \psi_k'' \psi_\ell'' \right\}$$

$$\times \left\{ \frac{b^2}{d^2} \sin^2 \theta + \cos^2 \theta \right\} d\eta$$

$$A5T_{ijk\ell} = \int_0^1 \psi_i \left\{ \psi_j^{iv} \psi_k \psi_\ell' + 4\psi_j \psi_k''' \psi_\ell'' + \psi_j'' \psi_k'' \psi_\ell' \right\} \left\{ \frac{b^2}{d^2} \sin^2 \theta + \cos^2 \theta \right\} d\eta$$

$$A6T_{ijk\ell} = \int_0^1 \psi_i \left\{ 2\psi_j'' \psi_k''' \psi_\ell' + 1.5\, \psi_j'' \psi_k'' \psi_\ell'' + 2\, \psi_j \psi_k''' \psi_\ell'' + 1.5\, \psi_j \psi_k'' \psi_\ell''' \right.$$

$$\left. + \psi_j \psi_k^{iv} \psi_\ell' + 0.25\, \psi_j \psi_k \psi_\ell^{iv} \right\} \sin 2\theta \, d\eta$$

$$A7T_{ijk\ell} = \int_0^1 \psi_i \left\{ 0.5\, \psi_j'' \psi_k'' \psi_\ell'' + 1.5\, \psi_j \psi_k'' \psi_\ell''' + 0.25\, \psi_j \psi_k \psi_\ell^{iv} \right\} \sin 2\theta \, d\eta$$

$$A8T_{ijk\ell} = \int_0^1 \psi_i \left\{ \psi_j'' \psi_k'' \psi_\ell'' + 4\, \psi_j \psi_k'' \psi_\ell''' + \psi_j \psi_k \psi_\ell^{iv} \right\} \left\{ \frac{b^2}{d^2} \cos^2 \theta + \sin^2 \theta \right\} d\eta$$

$$A9T_{ijk\ell} = \int_0^1 \psi_i \left\{ \psi_j'' \psi_k'' \psi_\ell'' + 3\psi_j \psi_k''' \psi_\ell'' + \psi_j'' \psi_k \psi_\ell''' + \psi_j \psi_k \psi_\ell^{iv} \right\}$$

$$\times \left\{ \frac{b^2}{d^2} \sin^2 \theta + \cos^2 \theta \right\} d\eta$$

$$B1T_{ijk\ell} = \int_0^1 \theta_i \theta_j \psi_k'' \psi_\ell'' \cos 2\theta \, d\eta$$

$$B2T_{ijk\ell} = \int_0^1 \theta_i \theta_j \psi_k'' \psi_\ell'' \sin 2\theta \, d\eta$$

$$C1T_{ijk\ell} = \int_0^1 \psi_i \left\{ \psi_j \psi_k \psi_\ell''' + \psi_j \psi_k'' \psi_\ell'' \right\} \cos 2\theta \, d\eta$$

$$C2T_{ijk\ell} = \int_0^1 \psi_i \left\{ \psi_j''' \theta_k \theta_\ell + 2\psi_j'' \theta_k \theta_\ell' \right\} \cos 2\theta \, d\eta$$

$$C3T_{ijk\ell} = \int_0^1 \psi_i \psi_j'' \theta_k \theta_\ell \cos 2\theta \, d\eta$$

$$C4T_{ijk\ell} = \int_0^1 \psi_i \psi_j' \psi_k' \psi_\ell'' \cos 2\theta \, d\eta$$

$$C5T_{ijk\ell} = \int_0^1 \psi_i \left\{ \psi_j''' \theta_k \theta_\ell + 2\psi_j'' \theta_k \theta_\ell' \right\} \sin 2\theta \, d\eta$$

$$C6T_{ijk\ell} = \int_0^1 \psi_i \left\{ \psi_j' \psi_k' \psi_\ell''' + \psi_j' \psi_k'' \psi_\ell'' + 0.5 \psi_j'' \psi_k' \psi_\ell'' \right\} \sin 2\theta \, d\eta$$

$$C7T_{ijk\ell} = \int_0^1 \psi_i \left\{ \psi_j''' \psi_k' \psi_\ell' + 1.5 \psi_j' \psi_k'' \psi_\ell'' \right\} \sin 2\theta \, d\eta$$

$$C8T_{ijk\ell} = \int_0^1 \psi_i \psi_j' \psi_k' \psi_\ell'' \sin 2\theta \, d\eta$$

$$C9T_{ijk\ell} = \int_0^1 \psi_i \psi_j' \theta_k \theta_\ell \sin 2\theta \, d\eta$$

$$D1T_{ijk\ell} = \int_0^1 \psi_i \left\{ \psi_j''' \theta_k' \theta_\ell' + 4\psi_j'' \theta_k' \theta_\ell'' + 2\psi_j' \theta_k'' \theta_\ell'' + 2\psi_j' \theta_k' \theta_\ell''' \right\} d\eta$$

$$D2T_{ijk\ell} = \int_0^1 \psi_i \left\{ 2\psi_j'' \psi_k'' \psi_\ell'' + 2\psi_j'' \psi_k' \psi_\ell''' + 4\psi_j''' \psi_k'' \psi_\ell' + \psi_j^{iv} \psi_k' \psi_\ell' \right\} d\eta$$

$$D3T_{ijk\ell} = \int_0^1 \psi_i \left\{ 4\psi_j' \psi_k'' \psi_\ell' + \psi_j' \psi_k' \psi_\ell'' \right\} \sin 2\theta \, d\eta$$

$$D4T_{ijk\ell} = \int_0^1 \psi_i \left\{ 3\psi_j'' \psi_k'' \psi_\ell' + 4\psi_j' \psi_k''' \psi_\ell' + 5\psi_j' \psi_k'' \psi_\ell'' + \psi_j' \psi_k' \psi_\ell''' \right\} \cos 2\theta \, d\eta$$

$$D5T_{ijk\ell} = \int_0^1 \psi_i \left\{ \psi_j'' \psi_k'' \psi_\ell'' + 2\psi_j' \psi_k'' \psi_\ell''' \right\} d\eta$$

$$D6T_{ijk\ell} = \int_0^1 \psi_i \left\{ \psi_j''' \theta_k' \theta_\ell' + 2\psi_j'' \theta_k' \theta_\ell'' \right\} d\eta$$

$$D7T_{ijk\ell} = \int_0^1 \theta_i \left\{ \theta_j' \psi_k'' \psi_\ell'' + \theta_j' \psi_k''' \psi_\ell' + \theta_j'' \psi_k'' \psi_\ell' \right\} d\eta$$

$$D8T_{ijk\ell} = \int_0^1 \psi_i \left\{ \psi_j' \psi_k' \psi_\ell''' + 2\psi_j' \psi_k'' \psi_\ell'' \right\} \cos 2\theta \, d\eta$$

$$UF_{ijk\ell} = \int_0^1 \psi_i \psi_j'' \left[\int_\eta^1 \int_0^{\bar{\eta}} (\psi_k' \psi_\ell' d\bar{\eta}) \, d\eta \right] dx - \int_0^1 \psi_i \psi_j' \int_0^\eta \psi_k' \psi_\ell' \, d\eta \, dx$$

$$T1_{ijk} = \int_0^1 \psi_i \left\{ \theta_j' \psi_k''' + 2\theta_j'' \psi_k'' + \theta_j''' \psi_k' \right\} d\eta$$

$$T4_{ijk} = \int_0^1 \psi_i \, (\theta_j'' \psi_k'' + \theta_j' \psi_k''') \, d\eta$$

$$T7_{ijk} = \int_0^1 \theta_i \theta_j' \theta_k'' \, d\eta$$

$$T8_{ijk} = \int_0^1 \theta_i \, (\psi_j' \psi_k''' + \psi_j'' \psi_k'') \, d\eta$$

The perturbation equations can now be written by inspecting the equilibrium equations presented above and by proceeding as in reference 10.

APPENDIX C - NOMENCLATURE

A	cross-sectional area of blade
$\{B\}$	vector
B_1	section constant
b,d	breadth and thickness of blade
d/b	thickness ratio
E	Young's modulus
$F1,F2...T3$	elements of matrix representing equilibrium equations (see appendix B)
$f_1,...f_{13}$	coefficients (see eq. (13))
G	modulus of rigidity
$I_{\eta\eta},I_{\xi\xi}$	area moments of inertia about major and minor principal centroidal axes
i,j,k,ℓ	dummy indices
J	torsional stiffness constant
k_A	blade cross-sectional polar radius of gyration
k_m	blade cross-sectional mass radius of gyration
L	length of blade
$l_1,l_2,...n_3$	direction cosines
m	mass of blade per unit length
n	number of nonrotating normal modes for each of the flapwise bending, edgewise bending, and torsional deflections
$O(\epsilon)$	small parameter of the order of bending slopes ($\simeq \overline{w}$, $\overline{v}$ or ϕ)
p	natural radian frequency
R	radius of disc
$\overline{r}_0,\overline{r}_1$	position vectors of a point on the blade elastic axis, before and after deformation, respectively
S,S_3	coordinate along the elastic axis before and after deformation, respectively
T	blade tension

t	time
u,v,w	displacements of arbitrary point on the elastic axis in x,y,z directions, respectively
u_F	foreshortening due to bending (see equation 5)
v_{oj},w_{oj},ϕ_{oj}	steady-state equilibrium quantities
$\bar{v},\bar{w},\phi$	steady-state dimensionless equilibrium displacements
x	running coordinate along x-axis
$x_oy_oz_o$	blade-fixed axis system at arbitrary point on elastic axis before deformation
XYZ	coordinate system with origin at hub (disc) center line which rotates with blade such that x-axis lies along the undeformed position of the elastic axis
xyz	blade-fixed axis system which translates with respect to $x_oy_oz_o$
y,z	centroidal principal axis of beam cross section
$x_3y_3z_3$	blade-fixed orthogonal axis system in deformed configuration obtained by rotating xyz; x_3 axis is tangent to the deformed elastic axis
α	setting angle (collective pitch)
β_{pc}	precone angle
γ	total pretwist of the blade over its length
$\gamma_{xx},\gamma_{x\eta},\gamma_{x\xi}$	engineering strain components
$\Delta v_j,\Delta w_j,\Delta\phi_j$	perturbation quantities
ε	extensional component of Green's strain tensor
ε_{ij}	Green's strain tensor
η	dimensionless length coordinate, x/L
$\hat{\eta},\hat{\xi}$	sectional coordinates along major andminor principal axes for a given point, respectively
θ	Eulerian-type rotation angle in appendix A, eq. (A12), where $\theta = \alpha + \gamma\eta + \phi = \theta_{pt} + \phi$
θ	geometric pitch angle in equations (1) to (16) and in appendix B ($\theta = \theta_{pt} = \alpha + \gamma\eta$)

$\theta_j(\eta)$	nonrotating torsional mode shape
$\lambda(\hat{\eta},\hat{\xi})$	warping function
λ_η,λ_ξ	derivatives of λ with respect to $\hat{\eta}$ and $\hat{\xi}$, respectively
λ_1	frequency parameter, $\sqrt{EI_{\eta\eta}/\rho AL^4}$
ξ	nondimensional rotational parameter, $EI_{\eta\eta}/\rho AL^4\Omega^2$
ρ	mass density
τ	dimensionless time, Ωt
ϕ	elastic twist about blade elastic axis
$\psi_j(\eta)$	nonrotating flatwise and edgewise bendign mode shapes
Ω	rotor blade angular velocity, rad/s
ω_1	exact fundamental mode frequency of straight, untwisted, nonrotating beam, 3.51602 λ_1
$\omega_{x3},\omega_{y3},\omega_{z3}$	torsional curvature (total rotation about x_3 axis) and bending curvatures, respectively
$(\)'$	primes denote differentiation with respect to x or η
$\dot{(\)}$	dot over a parameter represents differentiation with respect to t or τ

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TABLE I. - CONVERGENCE PATTERN OF FREQUENCY RATIOS (P/λ_1) OF PRETWISTED, PRECONED, ROTATING BLADE INCLUDING THIRD DEGREE GEOMETRIC NONLINEARITIES AND CORIOLIS EFFECTS

$\left[\frac{d}{b} = 0.05,\ \beta_{pc} = 15^\circ,\ \gamma = 30^\circ,\ \alpha = 0^\circ,\ \bar{R} = 0,\ \frac{L}{d} = 200,\ \frac{\Omega}{\omega_1} = 0.50.\right]$

Mode number	Results from perturbation solution: Galerkin method (Third degree equations)								Second degree equations (Ref. 10) n = 8	MSC/NASTRAN (500 CQUAD4 elements)
	†n = 1	n = 2	n = 3	n = 4	n = 5	n = 6	n = 7	n = 8		
1	6.7837	4.0123	3.9880	3.9820	3.9813	3.9802	3.9804	3.9805	3.9699	3.9923
2	65.0281	40.0635	20.4535	20.1407	20.0567	20.0226	20.0116	20.0073	19.9823	20.0707
3	74.0808	63.7502	60.0351	57.6998	57.2708	57.0335	56.9803	56.9702	51.6155	57.7069
4	-------	89.2954	85.0776	63.6660	63.0034	62.9864	62.9480	62.9221	60.1018	64.5911
5	-------	216.1802	137.8281	86.8464	86.3469	86.3378	86.2964	86.2841	85.5334	86.1357
6	-------	438.2073	214.4627	214.0674	124.1565	121.6039	121.2614	120.9963	120.8147	121.1038
7	-------	--------	351.9987	275.0052	214.0269	204.4178	200.0460	199.5745	199.2196	199.4268
8	-------	--------	457.2028	350.2601	249.5374	214.1014	213.9084	213.9329	204.6820	219.5793

†n represents the number of nonrotating normal modes used in the generalized coordinates.

TABLE II. - CONVERGENCE PATTERN OF STEADY STATE TIP DEFLECTIONS FROM BEAM THEORY INCLUDING THIRD DEGREE GEOMETRIC NONLINEARITIES, AND COMPARISON WITH MSC/NASTRAN RESULTS

$\left[\frac{d}{b} = 0.05,\ \beta_{pc} = 15^\circ,\ \gamma = 30^\circ,\ \alpha = 0^\circ,\ \bar{R} = 0,\ \frac{\Omega}{\omega_1} = 0.50.\right]$

Method	Number of assumed modes or CQUAD4 elements	Steady state tip deflection (centerline deflection)		
		$\bar{w}$	$\bar{v}$	ϕ
Galerkin	n = 1	-0.018444	0.00182864	-0.00429201
	n = 2	-.0547506	.00661266	-.00088087
	n = 3	-.0545014	.00585071	-.00086910
	n = 4	-.0547382	.00587672	-.00081777
	n = 5	-.0547296	.00584699	-.00081642
	n = 6	-.0547554	.00584616	-.00083679
	n = 7	-.0547519	.00584231	-.00083540
	n = 8	-.0547560	.00584234	-.000831394
MSC/ NASTRAN	500 CQUAD4 Elements	-.0543090	.00583720	-.000902900

TABLE III. - COMPARISON OF THEORETICAL AND EXPERIMENTAL RESULTS

$\left[\frac{d}{b} = 0.05, \frac{L}{d} = 60, \gamma = 0^\circ, L = 152.4 \text{ mm } (6.0 \text{ in}), E = 222.7 \text{ GPa}, G = 95.1477 \text{ GPa}, \rho = 2.02728 \times 10^{-8} \text{ kg/mm}^3\right]$

Precone angle β_{pc}, degree	Setting angle α, degree	Rotational speed Ω, rpm	Method	Natural frequency, Hz				
				Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
0	30	3600	(a)	126.4	582.2	623.9	1683.4	1825.8
			(b)	126.5	576.2	625.7	1689.3	1813.0
			(c)	129.0	570.0	630.0	1688.0	1807.0
		4800	(a)	146.5	588.7	647.0	1700.4	1837.4
			(b)	146.6	578.4	651.9	1717.0	1815.8
			(c)	146.0	582.0	665.0	1716.0	1817.0
		6000	(a)	168.6	597.0	673.4	1714.1	1853.3
			(b)	168.9	581.5	684.0	1751.3	1819.9
			(c)	166.0	587.0	706.0	------	------
22.5	60	1200	(a)	96.5	572.5	593.4	1655.3	1796.3
			(b)	96.5	573.3	593.6	1656.2	1796.2
			(c)	97.0	574.0	-----	------	1798.0
		3600	(a)	111.5	489.9	625.2	1569.0	1684.5
			(b)	114.4	563.8	622.9	1625.7	1696.7
			(c)	124.4	561.0	626.0	-----	------
45	0	870	(a)	94.8	570.2	591.0	1653.1	1780.5
			(b)	94.9	573.7	591.3	1653.9	1783.6
			(c)	95.9	574.0	588.0	1655.0	1799.0
		1506	(a)	96.3	541.0	593.0	1655.0	1675.0
			(b)	96.5	572.7	593.8	1657.3	1696.8
			(c)	98.6	569.0	589.0	1662.0	1803.0
22.5	90	2400	(a)	101.4	574.3	603.5	1665.4	1815.8
			(b)	101.5	574.1	603.7	1666.0	1815.3
			(c)	101.9	571.4	602.9	1664.0	1799.0
		4800	(a)	120.1	573.3	644.2	1700.6	1836.1
			(b)	120.0	573.2	644.4	1701.2	1835.7
			(c)	121.2	562.0	654.0	1684.0	1826.0
		6000	(a)	131.8	572.3	673.3	1721.9	1855.1
			(b)	131.7	572.2	673.4	1722.5	1854.7
			(c)	134.5	553.0	688.0	1695.0	1854.0

(a) Theoretical results excluding third degree geometric nonlinearities (Ref. 10).
(b) Present theoretical results incuding third degree geometric nonlinearities.
(c) Experimental results (Reference 10).

TABLE IV(a). - COMPARISON OF FREQUENCY RATIOS OF PRECONED, ROTATING BLADES OF LOW THICKNESS RATIO

$[\alpha = 0°, \bar{R} = 0, \frac{L}{b} = 10.]$

$\frac{d}{b}$	γ, degree	β_{pc}, degree	$\frac{\Omega}{\omega_1}$	Method	Frequency ratio, p/λ_1				
					Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
0.05	0	15	1.0	(a)	unstable	-------	-------	-------	-------
				(b)	5.1562	23.9231	42.5942	64.0009	90.3709
				(c)	5.1781	23.7926	44.2514	63.4078	88.2218
			2.0	(a)	unstable	-------	-------	-------	-------
				(b)	8.2498	28.5806	37.0869	69.5140	89.9382
				(c)	8.3385	28.3445	39.0512	68.0883	87.6030
0.05	30	15	0.8	(a)	unstable	-------	-------	-------	-------
				(b)	4.6283	20.4636	50.6977	61.8649	91.1574
				(c)	4.6363	20.5566	52.6995	61.7111	91.3101
			1.0	(a)	unstable	-------	-------	-------	-------
				(b)	5.1485	20.7221	47.2311	62.3392	92.4586
				(c)	5.1656	20.9982	49.2750	61.8346	92.8652
0.05	30	45	0.3	(a)	3.5517	19.7350	54.1394	60.3105	84.9439
				(b)	3.5652	19.7493	57.4509	63.2751	85.8371
				(c)	3.5762	19.7675	57.6886	65.1148	85.5516
			0.5	(a)	Unstable	-------	-------	-------	-------
				(b)	3.6965	19.3873	46.4173	60.6038	94.1009
				(c)	3.7167	19.7195	48.1580	60.3879	94.5271
0.06	0	45	0.3	(a)	3.5438	22.1084	50.2454	61.7771	72.2320
				(b)	3.5541	22.1433	52.2279	61.8699	74.9998
				(c)	3.5665	22.1613	52.9289	61.8449	74.6194
			0.5	(a)	Unstable	-------	-------	-------	-------
				(b)	3.7123	22.4986	39.1497	62.5760	86.6498
				(c)	3.7055	22.2822	40.7961	61.7824	85.4823
			1.0	(a)	Unstable	-------	-------	-------	-------
				(b)	4.8283	20.6450	24.8709	67.6783	94.9696
				(c)	4.8169	22.2298	23.0175	61.1692	94.4257

(a) Beam theory including second degree geometric nonlinearities.
(b) Beam theory including third dgree geometric nonlinearities.
(c) Results from MSC/NASTRAN.

TABLE IV(b). - COMPARISON OF FREQUENCY RATIOS OF PRECONED, ROTATING BLADES OF HIGH THICKNESS RATIO

$[\alpha = 0^\circ, \bar{R} = 0, \frac{L}{b} = 10.]$

$\frac{d}{b}$	γ, degree	β_{pc}, degree	$\frac{\Omega}{\omega_1}$	Method	Frequency ratio, p/λ_1				
					Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
0.10	30	15	2.0	(a)	8.2940	19.6256	29.0720	64.0094	72.7573
				(b)	8.2418	22.1184	35.5968	67.5836	78.0812
				(c)	8.3177	22.8858	35.5153	64.3812	77.5158
0.20	30	15	1.0	(a)	5.1411	15.4978	26.0439	60.8242	67.1784
				(b)	5.1538	15.5377	26.3139	61.3437	69.0699
				(c)	5.1619	15.5424	26.0745	60.7458	70.4579
0.20	30	45	1.0	(a)	0.1533	7.2997	19.0724	54.6244	63.4657
				(b)	4.8045	13.1377	26.5609	62.3999	75.6008
				(c)†	4.8128	13.5546	24.5559	58.1445	74.3679
0.25	0	45	1.0	(a)	4.6184	10.6218	23.2097	59.4123	62.8856
				(b)	4.8283	12.2162	24.8709	67.6783	68.7724
				(c)	4.8122	12.3085	23.0257	61.0894	67.1483
0.25	30	45	1.0	(a)	4.5877	10.5075	23.2236	56.6582	63.7352
				(b)	4.8175	11.7024	25.9483	61.1655	71.9986
				(c)	4.8046	11.8666	23.9928	57.2734	70.3849

(a)Beam theory including second degree geometric nonlinearities.
(b)Beam theory including third dgree geometric nonlinearities.
(c)Results from MSC/NASTRAN.
†Spurious instability was shown by MSC/NASTRAN in reference 10 due to inadequate inplane fixing stiffness parameter value.

TABLE V. – EFFECT OF IGNORING CERTAIN GEOMETRIC NONLINEAR TERMS ON THE FREQUENCY RATIOS AND STEADY STATE DEFLECTIONS OF ROTATING, PRECONED BLADES

$[\alpha = 0^\circ, \bar{R} = 0, \frac{L}{b} = 10.]$

$\frac{d}{b}$	$\frac{\Omega}{\omega_1}$	γ, degree	β_{pc}, degree	Mode number or tip deflection	MSC/NASTRAN	Nondimensional frequency parameter, p/λ_1, ignoring Coriolis effects						
						Third degree geometric nonlinear equations						Second degree equations
						Case (1)	Case (2)	Case (3)	Case (4)	Case (5)	Case (6)	
0.05	1.0	30	15	Mode 1	5.1656	5.1485	5.1812	5.1808	5.1807	5.1806	7.0576	Unstable
				Mode 2	20.9982	20.7221	20.7477	20.7423	20.7420	20.7130	22.5000	--------
				Mode 3	49.2750	47.2311	47.3168	47.3144	47.3102	46.5993	51.5012	--------
				Mode 4	61.8346	62.3392	62.3617	62.3509	62.3515	62.2591	63.1529	--------
				Mode 5	92.8652	92.4586	92.4044	92.4058	92.4084	91.0954	89.9055	--------
				$\bar{w}$	-0.130684	-0.132132	-0.131552	-.0131559	-0.131562	-0.131558	-0.105646	--------
				$\bar{v}$	0.012878	0.013097	0.013022	0.013024	0.013025	0.013023	0.010946	--------
				ϕ	-0.004148	-0.001778	-0.001812	-0.001808	-0.001808	-0.001847	-0.000827	--------
0.20	1.0	30	45	Mode 1	4.8128	4.8045	4.9297	4.9265	4.9265	4.9264	5.8022	0.1533
				Mode 2	13.5546	13.1377	13.2568	13.2127	13.2123	13.2066	13.4365	7.2997
				Mode 3	24.5559	26.5609	25.5728	26.5587	26.5585	26.5568	27.8104	19.0724
				Mode 4	58.1445	62.3999	62.4503	62.3605	62.3606	62.3526	61.9055	54.6244
				Mode 5	74.3679	75.6008	75.3623	75.3583	75.3590	75.2874	74.1130	63.4567
				$\bar{w}$	-0.357180	-0.378955	-0.371073	-0.371247	-0.371248	-0.371247	-0.321789	-0.398513
				$\bar{v}$	0.035096	0.038377	0.037235	0.037386	0.037386	0.037386	0.037118	0.028370
				ϕ	-0.009450	-0.006834	-0.007093	-0.006946	-0.006945	-0.006942	-0.000297	-0.069898
0.10	2.0	0	15	Mode 1	8.3327	8.2498	8.4302	8.4302	8.4302	8.4302	8.4302	8.3781
				Mode 2	28.3285	27.8202	28.0177	28.0152	28.0152	28.0152	28.0152	21.3722
				Mode 3	28.7811	28.5806	28.7117	28.7117	28.7117	28.7117	28.7117	28.3088
				Mode 4	68.0478	69.5140	69.6073	69.6073	69.6073	69.6073	69.6073	68.3325
				Mode 5	75.6988	77.7080	77.4843	77.4841	77.4841	77.4841	77.4841	70.3606
				$\bar{w}$	-0.200225	-0.205344	-0.202174	-0.202174	-0.202174	-0.202174	-0.202174	-0.203129
				$\bar{v}$	---------	---------	---------	---------	---------	---------	---------	---------
				ϕ	---------	---------	---------	---------	---------	---------	---------	---------

Case (1): Full third degree geometric nonlinear equations considered.
Case (2): Third degree nonlinear terms arising from (TW')' and (TV')' are ignored.
Case (3): Third degree nonlinear terms from (TW')', (TV')' and also those from shearing strains are ignored.
Case (4): In addition to those in case (3), third degree nonlinear terms associted with EB_1 in the bending equations are ignored.
Case (5): In addition to those in case (3), third degree nonlinear terms associated with EB_1 are ignored in all equations.
Case (6): All third degree nonlinear terms are ignored in the bending equations but retained in the torsion equation. Further, all EB_1 terms are discarded in all equations.

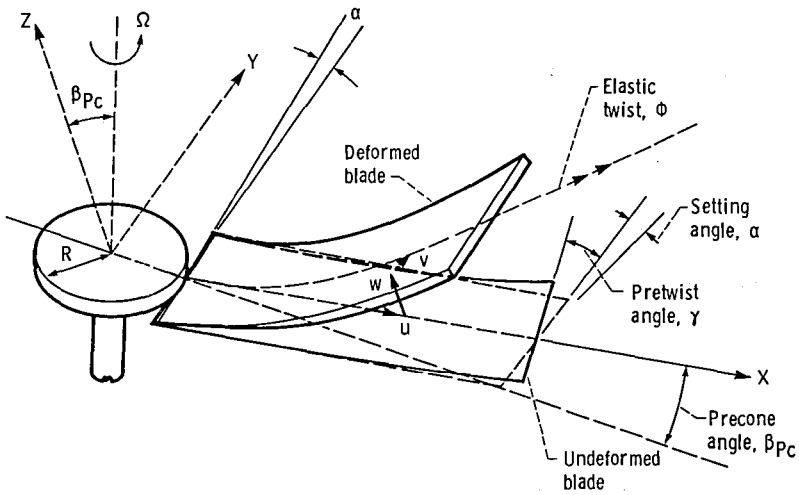


Figure 1. - Blade coordinate system and definition of blade parameters.

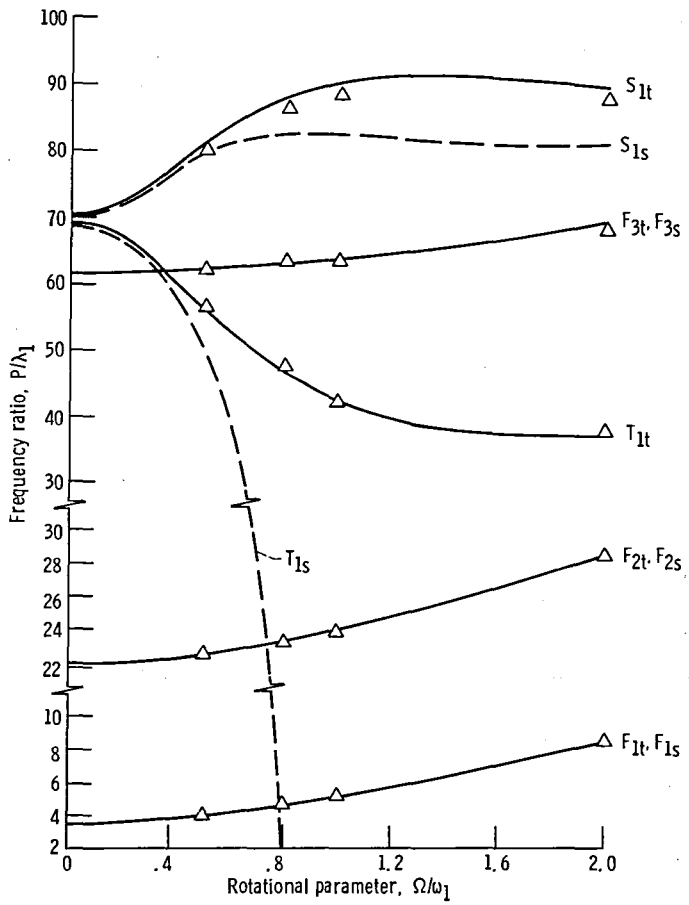


Figure 2. - Comparison of frequency ratios predicted by various methods $\left(\frac{d}{b} = 0.05,\ \alpha = \Phi = 0^0,\ \beta_{Pc} = 15^0,\ \bar{R} = 0\right)$

F_1, F_2, F_3 are flatwise modes (subscript s denotes solution of second degree equations and t denotes solution of third degree equations), and T_1 and S_1 are first torsional and edgewise bending modes, respectively; △ indicates the result produced by MSC/NASTRAN.

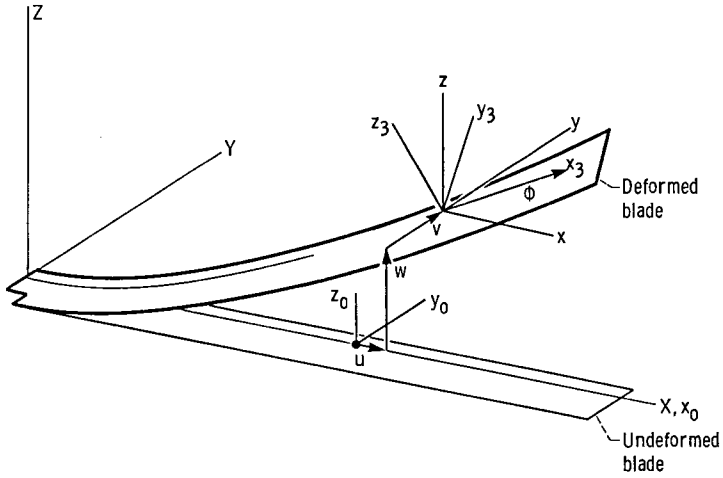


Figure 3. - Schematic representation of undeformed and deformed blade for flap-lag-pitch rotational transformation sequence.

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15. Supplementary Notes

K.B. Subrahmanyam, on leave from NBKR Institute of Science and Technology, mechanical Engineering Dept., Vidyanagar 524413, India and presently Senior Research Associate, University of Toledo, Mechanical Engineering Dept., Toledo, Ohio 43606; K.R.V. Kaza, NASA Lewis Research Center.

16. Abstract

This report is concerned with the study of the influence of third degree geometric nonlinear terms on the vibration and stability characteristics of rotating, pretwisted, and preconed blades. The governing coupled flapwise bending, edgewise bending, and torsional equations are derived including third-degree geometric nonlinear elastic terms by making use of the geometric nonlinear theory of elasticity in which the elongations and shears are negligible compared to unity. These equations are specialized for blades of doubly symmetric cross section with linear variation of pretwist over the blade length. The nonlinear steady state equations and the linearized perturbation equations are solved by using the Galerkin method, and by utilizing the nonrotating normal modes for the shape functions. Parametric results obtained for various cases of rotating blades from the present theoretical formulation are compared to those produced from the finite element code MSC/NASTRAN, and also to those produced from an in-house experimental test rig. It is shown that the spurious instabilities, observed for thin, rotating blades when second degree geometric nonlinearities are used, can be eliminated by including the third-degree elastic nonlinear terms. Furthermore, inclusion of third degree terms improves the correlation between the theory and experiment.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
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